2011/04/25 PDEセミナー

「退化 Keller-Segel 方程式系の解の漸近挙動とHölder評価」

§ Keller-Segel system

$$
(\mathrm{dKS})\begin{cases}
\partial_t u-\Delta u^{\alpha}+\mathrm{div}(u\nabla\psi)=0, & t>0,\ x\in\mathbb{R}^n\\
-\Delta\psi+\psi=u & \quad ''\\
u(0,x)=u_0(x)\geq 0, & x\in\mathbb{R}^n
\end{cases}
$$

$$
n\geq 3,\quad 1<\alpha\leq 2-\frac{2}{n},\quad u_0\in C_0^{\infty}(\mathbb{R}^n)
$$

$$
\Delta u^{\alpha}=\alpha\,\mathrm{div}(\underbrace{u^{\alpha-1}}_{\substack{\shortparallel\\ 0}}\nabla u)\quad \text{degenerate}
$$

when $u=0$

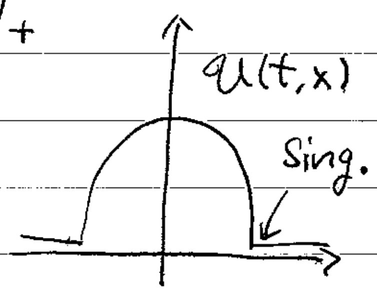
⟨porous medium eq.⟩

$$
\partial_t w-\Delta w^{\alpha}=0
$$

⟨Barenblatt sol.⟩

$$
\mathcal{U}(t,x):=(1+\sigma t)^{-\frac{n}{\sigma}}\left(A-\frac{|x|^2}{(1+\sigma t)^{\frac{2}{\sigma}}}\right)_+^{\frac{1}{\alpha-1}}
$$

where $\sigma=n(\alpha-1)+2$, $A>0$



Def.

(u,ψ) : weak sol. of (dKS)

$\underset{\text{def}}{\Longleftrightarrow}$ $\exists T>0$ s.t.

- $u\geq 0$ a.e. $(0,T)\times\mathbb{R}^n$
- $u\in L^{\infty}(0,T;L^1\cap L^{\alpha})$, $\nabla u^{\alpha-\frac{1}{2}}\in L^2((0,T)\times\mathbb{R}^n)$
- u satisfies (dKS) in distribution.

$\psi=(-\Delta+1)^{-1}u.$

Prop. (Sugiyama-Kunii '06)

$1 < \alpha \le 2 - \frac{2}{n}$, u_0 : small

$\Rightarrow$ $\exists$ weak sol. of (dKS), $T = \infty$.

$$u(t) \to 0, \quad t \to \infty$$

Problem

Asymp. behavior of u as $t \to \infty$

⟨Known Results⟩ $\|u_0\|_1 = \|\mathcal{U}(0)\|_1$

- Luckhaus-Sugiyama '07

  $1 < \alpha \le 2 - \frac{2}{n} \Rightarrow \|u(t) - \mathcal{U}(t)\|_1 \to 0 \quad t \to \infty$.

- Ogawa '08

  $1 < \alpha < 2 - \frac{2}{n} \Rightarrow \|u(t) - \mathcal{U}(t)\|_1 \le C(1+t)^{-\nu}$

  for some $\nu > 0$.

Thm. (Ogawa-M.)

$\alpha = 2 - \frac{2}{n} \Rightarrow \|u(t) - \mathcal{U}(t)\|_1 \le C(1+t)^{-\nu}$

ν depends on "reg. of u", n, u_0.

§ formal proof.

⟨forward self-sim. trans.⟩

$$s = \frac{1}{\sigma}\log(1+\sigma t), \quad y = \frac{x}{(1+\sigma t)^{\frac{1}{\sigma}}}$$

$$v(s, y) := (1+\sigma t)^{\frac{n}{\sigma}} u(t, x)$$

$$\phi(s, y) := \quad " \quad \psi(t, x)$$

$$\Rightarrow \begin{cases} \partial_s v = \operatorname{div}\Big(v \nabla \big(\underbrace{\tfrac{\alpha}{\alpha-1} v^{\alpha-1} + \tfrac{1}{2}|y|^2}_{\downarrow\, s\to\infty \atop A} - \underbrace{e^{-\kappa s}\phi}_{\downarrow\, s\to\infty \atop 0} \big) \Big) \\ (-e^{-2s}\Delta + 1)^{-1}\phi = v \\ v(0,y) = u_0(y) \end{cases}$$

i.e. $v^{\alpha-1} \to \left(A - \frac{\alpha-1}{2\alpha}|y|^2\right)_+$

where $\kappa = (2n-n) + 2 - \sigma = n - n(\alpha-1)$

$= n(2-\alpha) \geq 2.$ $\leftarrow \alpha = 2 - \frac{2}{n}.$

$$-e^{-\kappa s}\operatorname{div}(v\nabla\phi) \simeq v \underbrace{\frac{-e^{-\kappa s}\Delta}{-e^{-2s}\Delta+1}}_{\text{not decay } (\kappa=2)} v$$

Lem.

v : unif. Hölder conti. on $(1,\infty)\times\mathbb{R}^n$.

$$\frac{-e^{-2s}\Delta}{-e^{-2s}\Delta+1} v \simeq \underbrace{e^{-\gamma s}}_{\text{decay!!}} \underbrace{\frac{e^{-(2-\gamma)s}|\nabla|^{2-\gamma}}{-e^{-2s}\Delta+1}}_{\text{b'dd}} \underbrace{|\nabla|^{\gamma} v}_{\text{b'dd (Hölder conti.)}}$$

§ Hölder estimate.

(PME) $\quad \partial_t u^{\frac{1}{\alpha}} - \Delta u = \operatorname{div} F$

$(F = xv - e^{-2t} v\nabla\phi, \quad u = v^{\alpha})$

⟨Known Results⟩

◎ Caffarelli-Friedman '80, DiBenedetto-Friedman '85

Hölder conti. of u.

Thm.

$F \in L^\infty(0,\infty : L^p)$, $p > n$

u : sol. of (PME), bdd.

$\Rightarrow$ $\exists C$, $\exists \gamma > 0$ s.t.

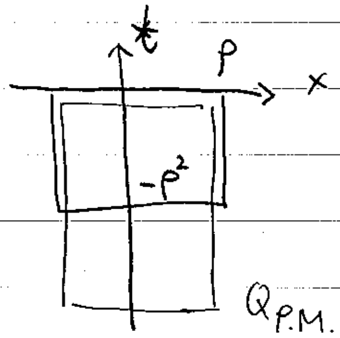
$$|u(t,x) - u(s,y)| \le C\left(\|u\|_\infty + \|F\|_{L^\infty(0,\infty : L^p)}\right) \times \left(\|u\|_\infty^{\frac{\gamma}{2}(1-\frac{1}{\alpha})} |t-s|^{\frac{\gamma}{2}} + |x-y|^\gamma\right)$$

⟨Scaling⟩

$\rho, M > 0$. $t = \dfrac{\rho^2}{M^{1-\frac{1}{\alpha}}} s$, $x = \rho y$

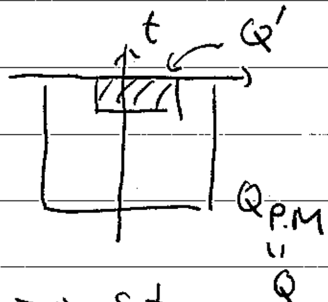
$u_{\rho,M}(s,y) = \dfrac{1}{M} u(t,x)$.

$$\partial_t u^{\frac{1}{\alpha}} - \Delta u = 0 \underset{\text{equiv.}}{\iff} \partial_s u_{\rho,M}^{\frac{1}{\alpha}} - \Delta u_{\rho,M} = 0$$



$$Q_{\rho,M} := \left(-\frac{\rho^2}{M^{1-\frac{1}{\alpha}}}, 0\right) \times B_\rho$$

Lem. (Alternative lem.)



$0 < \rho \ll 1$, $M \simeq \sup_{Q_{\rho,M}} u$

$\Rightarrow$ $0 < \exists \theta_0$, $\exists \eta_0 < 1$, $\exists Q' \subset\subset Q_{\rho,M} = Q$, $\exists \lambda_0 > 0$ s.t.

- (lower bounds)

$$\frac{|Q \cap \{u < \lambda_0\}|}{|Q|} \le \theta_0 \Rightarrow \inf_{Q'} u \ge \inf_Q u + \eta_0 \operatorname{osc}_Q u$$

- (upper bounds)

$$〃 > \theta_0 \Rightarrow \sup_{Q'} u \le \sup_Q u - \eta_0 \operatorname{osc}_Q u$$

$$\therefore \ \operatorname*{osc}_{Q'} u = \sup_{Q'} u - \inf_{Q'} u \leq (1-\eta_0) \operatorname*{osc}_{Q} u$$

$\exists \{Q_m\}_{m=0}^{\infty}$, $Q_0 \supset Q_1 \supset \cdots$, $\exists\, 0 < r_0 < 1$, $\rho_m = \operatorname{diam} Q_m$ s.t.

$$\rho_m = r_0 \rho_{m-1} = \cdots = r_0^m \rho_0$$

$$\operatorname*{osc}_{Q_m} u \leq (1-\eta_0) \operatorname*{osc}_{Q_{m-1}} u \qquad \Longleftrightarrow \frac{\rho_m}{\rho_0} = r_0^m$$

$$\leq \cdots \leq (1-\eta_0)^m \operatorname*{osc}_{Q_0} u$$

$$\lesssim C \left(\frac{\rho_m}{\rho_0} \right)^{\gamma} \qquad \gamma = \frac{\log(1-\eta_0)}{\log r_0} > 0$$

<Summery>

$\operatorname*{osc}_{Q_0} u$

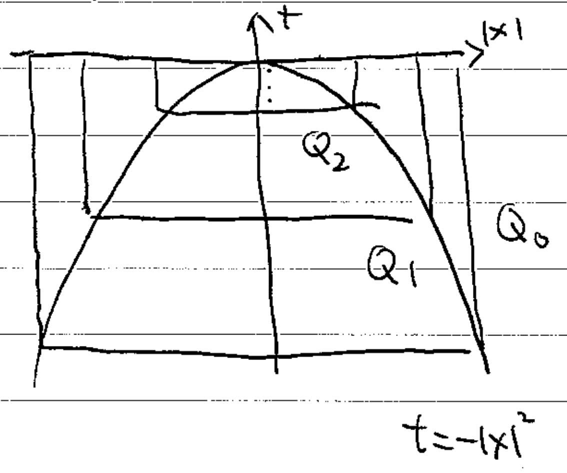
lower bounds　　Alt. Lem　　upper bounds

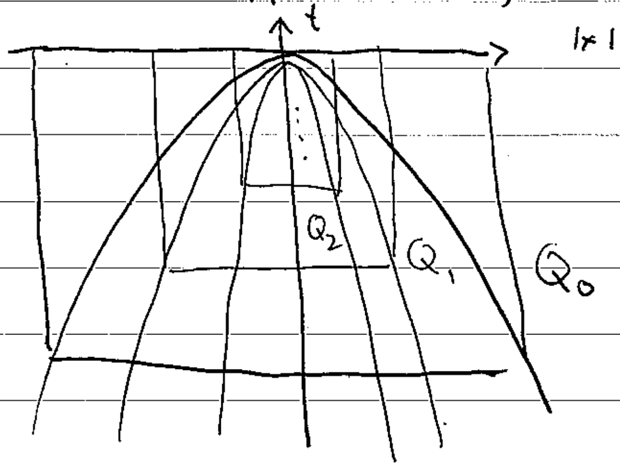
$$\operatorname*{osc}_{Q_1} u \leq (1-\eta_0) \operatorname*{osc}_{Q_0} u$$

$$\Downarrow$$

$$\operatorname*{osc}_{Q_m} u \leq (1-\eta_0)^m \operatorname*{osc}_{Q_0} u$$

$Q_\rho = (-\rho^2, 0) \times B_\rho$　　　　$Q_{\rho,M} = \left(-\frac{\rho^2}{M^{1-\frac{1}{\alpha}}}, 0\right) \times B_\rho$





pf. of lower bounds

Lem. (Caccioppoli est.)

$\forall k>0.\ \exists C>0$ s.t. $Q = I \times B$

$$\frac{1}{M^{1-\frac{1}{\alpha}}} \sup_I \int_B (u(t)-k)_-^2\,dx + \iint_Q |\nabla(u-k)_-|^2\,dt\,dx$$

(CE)

$$\leq C\|F\|^2_{L^\infty(L^p)} \int_I |\{u(t)<k\}|^{1-\frac{2}{p}}\,dt + \text{error}.$$

Outline of pf. $\quad (u-k)_- \sim k-u$

$-\iint_Q (\text{PME}) \times (u-k)_-\,dt\,dx$

$$\Rightarrow\ -\iint_Q \underbrace{\partial_t u^{\frac{1}{\alpha}}}_{\shortparallel\ -\frac{1}{\alpha}u^{\frac{1}{\alpha}-1}\partial_t(u-k)_-}(u-k)_-\,dt\,dx - \iint_Q \underbrace{\nabla u}_{\shortparallel\ -\nabla(u-k)_-}\cdot\nabla(u-k)_-\,dt\,dx$$

$$= \iint_Q F\cdot\nabla(u-k)_-\,dt\,dx \leq \frac{1}{4}\iint_Q |\nabla(u-k)_-|^2\,dt\,dx + \iint_{Q\cap\{u>k\}} |F|^2\,dt\,dx$$

$$\iint_{Q\cap\{u>k\}} |F|^2\,dt\,dx \leq \int_I \underbrace{\int_{\{u(t)>k\}} |F|^2\,dx}_{\leq \|F\|^2_{L^p}|\{u(t)>k\}|^{1-\frac{2}{p}}}\,dt$$

□ (CE)

$$\|(u-k)_-\|_{L^{2+\frac{4}{n}}} \leq (\text{RHS of (CE)})$$

$$\to 0 \quad k \downarrow k'$$

$\therefore\ (u-k')_- = 0$ in $Q' \subset\subset Q$

i.e. $u \geq k'$

□ (lower bounds)