2011/02/16-18 若手のための偏微分方程式と数学解析

退化 Keller-Segel 系の解の正則性と漸近挙動

§ degenerate Keller-Segel system.

$$(\mathrm{dKS})\begin{cases}\partial_t u-\Delta u^{\alpha}+\mathrm{div}\,(u\nabla\psi)=0, & t>0,\ x\in\mathbb{R}^n\\ -\Delta\psi+\psi=u, & \text{〃}\\ u(0,x)=u_0(x)\geq 0 & \end{cases}$$

$n\geq 3,\quad 1<\alpha\leq 2-\frac{2}{n},\quad u_0\in C_0^{\infty}.$

$$-\Delta u^{\alpha}=-\mathrm{div}(\underbrace{\alpha u^{\alpha-1}}_{=0\ \text{if}\ u=0}\nabla u)\quad \text{degenerate}.$$

⟨porous medium eq.⟩

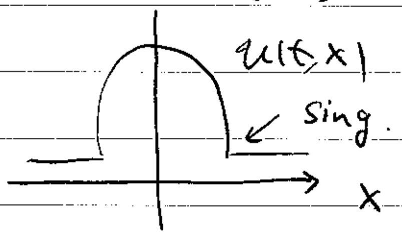
$$\partial_t w-\Delta w^{\alpha}=0.$$

⟨Barenblatt sol.⟩

$$\mathcal{U}(t,x)=(1+\sigma t)^{-\frac{n}{\sigma}}\left(A-\frac{\alpha-1}{2\alpha}\frac{|x|^2}{(1+\sigma t)^{\frac{2}{\sigma}}}\right)_+^{\frac{1}{\alpha-1}}$$

$\sigma=n(\alpha-1)+2$

$A>0.$



Def.

u: weak sol. of (dKS)

$\underset{\text{def}}{\Longleftrightarrow}\ {}^{\exists}T>0$ s.t.

(1) $u\geq 0$ a.e. $(0,T)\times\mathbb{R}^n$

(2) $u\in L^{\infty}(0,T;L^{\alpha-1}),\ \nabla u^{\alpha-\frac{1}{2}}\in L^2((0,T)\times\mathbb{R}^n)$

(3) u satisfies (dKS) in distribution

where $\psi=(-\Delta+1)^{-1}u$

Prop. (Sugiyama-Kunii '06)

u_0 : small $\Longrightarrow$ $^\exists u$: global sol. of (dKS)

$u(t) \to 0$ as $t \to \infty$.

Problem

Asymp. behavior of u as $t \to \infty$.

⟨Known Results⟩ $\|\mathcal{U}(0)\|_1 = \|u_0\|_1$

- Luckhaus-Sugiyama '07

$1 < \alpha \leq 2 - \frac{2}{n} \Longrightarrow \|u(t) - \mathcal{U}(t)\|_1 \to 0 \quad t \to \infty$.

- Ogawa '08

$1 < \alpha < 2 - \frac{2}{n} \Longrightarrow \|u(t) - \mathcal{U}(t)\|_1 \leq {}^\exists C(1+t)^{-\nu}$

Thm. (Ogawa-M.)

$\alpha = 2 - \frac{2}{n} \Longrightarrow {}^\exists C, \nu > 0$ s.t.

$\|u(t) - \mathcal{U}(t)\|_1 \leq C(1+t)^{-\nu}$

ν : dep. on "reg. of u".

§ formal proof $\alpha = 2 - \frac{2}{n}$.

⟨forward self-sim. trans⟩

$$s = \frac{1}{\sigma}\log(1+\sigma t), \quad y = \frac{x}{(1+\sigma t)^{\frac{1}{\sigma}}},$$

$$v(s, y) = (1+\sigma t)^{\frac{n}{\sigma}} u(t, x)$$
$$\phi(s, y) = (1+\sigma t)^{\frac{n}{\sigma}} \psi(t, x)$$

$$\Rightarrow \begin{cases} \partial_s v = \operatorname{div}\Big(v \nabla\big(\underbrace{\tfrac{\alpha}{\alpha-1} v^{\alpha-1} + \tfrac{1}{2}|y|^2}_{\downarrow \, A} \underbrace{- e^{-2s}\phi}_{\downarrow \, 0}\big)\Big) \quad s\to\infty \\ -e^{-2s}\Delta\phi + \phi = v \\ v(0, y) = v_0(y) \end{cases}$$

i.e. $v^{\alpha-1} \to \left(A - \frac{\alpha-1}{2\alpha}|y|^2\right)_+$

$$-e^{-2s}\operatorname{div}(v\nabla\phi) \simeq v \underbrace{\frac{-e^{-2s}\Delta}{-e^{-2s}\Delta+1}}_{\text{bdd}} v$$

bdd. $\left(\begin{array}{l} \alpha < 2 - \frac{2}{n} \\ \Rightarrow \text{decay} \end{array}\right)$

<u>Lem.</u>

v = unif. Hölder conti. on $(1,\infty)\times\mathbb{R}^n$

$$\frac{-e^{-2s}\Delta}{-e^{-2s}\Delta+1} v \simeq \underbrace{e^{-\gamma s}}_{\text{decay!!}} \frac{e^{-(2-\gamma)s}|\nabla|^{2-\gamma}}{e^{-2s}|\nabla|^2+1} \underbrace{|\nabla|^{\gamma} v}_{\text{bdd (Hölder conti)}}$$

§ Hölder est.

(PME) $\partial_t u^{\frac{1}{\alpha}} - \Delta u = \operatorname{div} F$

$$F = xv - e^{-2t} v \nabla \phi$$
$$(x \leftarrow y,\ t \leftarrow s,\ u = v^{\alpha})$$

Thm.

$p > n \Rightarrow \exists C, \exists \gamma > 0$ s.t.

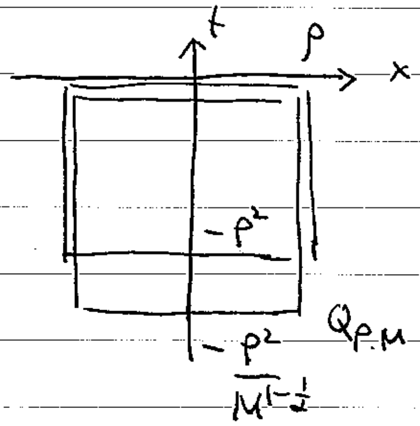
$$|u(t,x) - u(s,y)| \le C \left(\|u\|_\infty + \|F\|_{L^\infty(L^p)} \right) \times \left(\|u\|_\infty^{\frac{\gamma}{2}(1-\frac{1}{\alpha})} |t-s|^{\frac{\gamma}{2}} + |x-y|^{\gamma} \right)$$

⟨Scaling⟩

$\rho, M > 0$. $t = \dfrac{\rho^2}{M^{1-\frac{1}{\alpha}}} s$, $x = \rho y$, $u_{\rho,M}(s,y) = \frac{1}{M} u(t,x)$

$$\partial_t u^{\frac{1}{\alpha}} - \Delta_x u = 0 \iff \partial_s u_{\rho,M}^{\frac{1}{\alpha}} - \Delta_y u_{\rho,M} = 0$$

$$Q_{\rho,M} := \left(-\frac{\rho^2}{M^{1-\frac{1}{\alpha}}}, 0 \right) \times B_\rho$$



Lem. (Alt. lem.)

$0<\rho\ll 1$, $M \gtrsim \sup_{Q_{\rho,M}} u$, $\cancel{\omega = \operatorname{osc}_{Q_{\rho,M}} u = \sup u - \inf u}$

$\Rightarrow$ $0 < {}^\exists\theta_0, {}^\exists\eta_0 < 1$, ${}^\exists\lambda_0 > 0$, ${}^\exists Q' \subset\subset Q_{\rho,M}$ s.t.

- (lower bounds)

$$\frac{|Q_{\rho,M}\cap\{u<\lambda_0\}|}{|Q_{\rho,M}|} < \theta_0 \Rightarrow \inf_{Q'} u \geq \inf_{Q_{\rho,M}} u + \eta_0 \underset{Q_{\rho,M}}{\overset{\sup-\inf}{\overset{\parallel}{\operatorname{osc}}}} u$$

- (upper bounds)

$$\text{〃} \quad \geq \theta_0 \Rightarrow \sup_{Q'} u \leq \sup_{Q_{\rho,M}} u - \eta_0 \operatorname{osc}_{Q_{\rho,M}} u .$$

In each cases

$$\operatorname{osc}_{Q'} u = \sup_{Q'} u - \inf_{Q'} u \leq (1-\eta_0)\operatorname{osc}_{Q_{\rho,M}} u .$$

$0< {}^\exists r_0 < 1$, ${}^\exists\{Q_j\}_{j=0}^{\infty}$, $\rho_j := \operatorname{diam} Q_j$ s.t.

$$\rho_j = r_0\rho_{j-1} = \cdots = r_0^j \rho_0 \Leftrightarrow r_0^j = \frac{\rho_j}{\rho_0}$$

$$\begin{aligned}\operatorname{osc}_{Q_j} u &\leq (1-\eta_0)\operatorname{osc}_{Q_{j-1}} u \\ &\leq \cdots \leq (1-\eta_0)^j \operatorname{osc}_{Q_0} u \\ &\leq C\rho_j^{\gamma} \qquad \gamma = \frac{\log(1-\eta_0)}{\log r_0}\end{aligned}$$

⟨Summary⟩

$$\underset{Q_0}{\operatorname{osc}}\, u$$

lower bounds Alt. Lem upper bounds

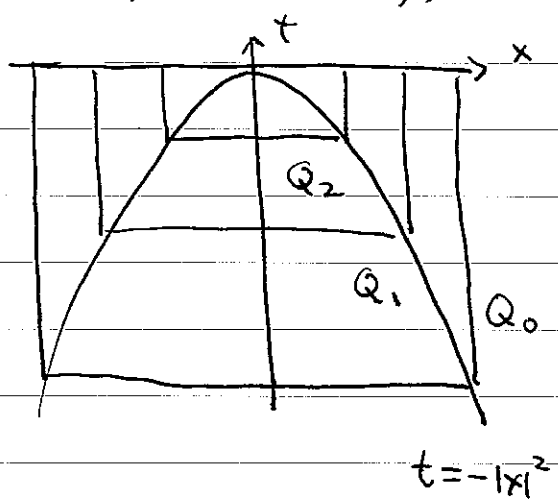
$$\underset{Q_1}{\operatorname{osc}}\, u \le (1-\eta_0)\, \underset{Q_0}{\operatorname{osc}}\, u$$

~~osc~~

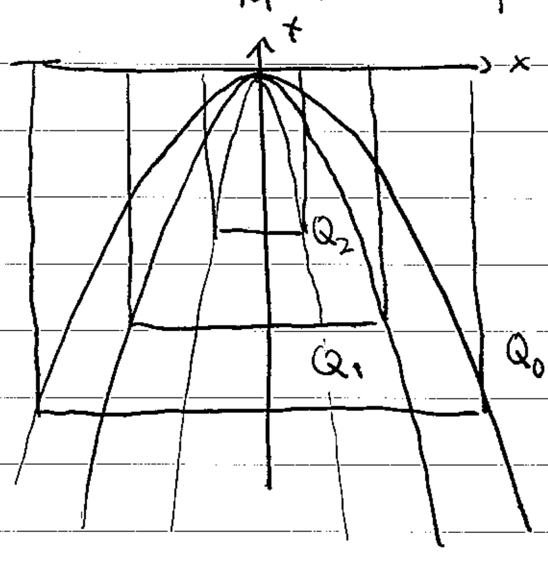
⋮

$$\underset{Q_j}{\operatorname{osc}}\, u \le (1-\eta_0)^j\, \underset{Q_0}{\operatorname{osc}}\, u$$

$Q_\rho = (-\rho^2, 0) \times B_\rho$



$Q_{\rho,M} = \left(-\frac{\rho^2}{M^{1-\frac{1}{a}}}, 0\right) \times B_\rho$



pf. of lower bounds

Lem (Caccioppoli est.)

$\forall k > 0.\ \exists C > 0$ s.t. $Q = I \times B$

(CE)
$$\frac{1}{M^{1-\frac{1}{\alpha}}} \sup_I \int_B (u(t)-k)_-^2 dx + \iint_Q |\nabla (u-k)_-|^2 dt dx$$
$$\leq C \|F\|_{L^\infty(L^p)}^2 \int_I |\{u(t) < k\}|^{1-\frac{2}{p}} dt + \text{error}$$

$\partial_t u^{\frac{1}{\alpha}} = \frac{1}{\alpha} u^{\frac{1}{\alpha}-1} \partial_t u$

key

$$- \iint_Q (\text{PME}) \times (u-k)_- \, dt dx$$

$$\|(u-k)_-\|_{L^{2+\frac{4}{n}}} \leq (\text{RHS of (CE)})$$

$\to 0 \qquad k \downarrow k'$

i.e. $(u-k')_- = 0$ in $Q' \subset\subset Q$

$\Updownarrow$

$u \geq k'$

$\square$ (lower bounds)