Weak mean curvature flow and boundary monotonicity formula for the Allen-Cahn equation

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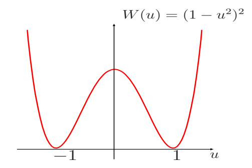
Allen-Cahn equation

$$\begin{cases} \varepsilon \dfrac{\partial u^\varepsilon}{\partial t} = \varepsilon \Delta u^\varepsilon - \dfrac{W'(u^\varepsilon)}{\varepsilon} = -\delta E^\varepsilon[u^\varepsilon] & t > 0,\ x \in \Omega, \\ \dfrac{\partial u^\varepsilon}{\partial \nu}\Big|_{\partial\Omega} = 0 & t > 0, \\ u^\varepsilon(0,x) = u_0^\varepsilon(x) & x \in \Omega. \end{cases} \tag{AC}$$

$\Omega \subset \mathbb{R}^n$: bounded domain, $\partial\Omega \in C^\infty$,
ν: outer unit normal on $\partial\Omega$,
$\varepsilon > 0$: small parameter
$W(u^\varepsilon) := (1-(u^\varepsilon)^2)^2$



$$E^\varepsilon[u^\varepsilon] := \int_\Omega \left(\frac{\varepsilon}{2}|\nabla u^\varepsilon|^2 + \frac{W(u^\varepsilon)}{\varepsilon} \right) dx =: \int_\Omega d\mu_t^\varepsilon.$$

Singular limiting problem

$$\begin{cases} \varepsilon \dfrac{\partial u^\varepsilon}{\partial t} = \varepsilon \Delta u^\varepsilon - \dfrac{W'(u^\varepsilon)}{\varepsilon} = -\delta E^\varepsilon[u^\varepsilon] & t > 0,\ x \in \Omega, \\ \dfrac{\partial u^\varepsilon}{\partial \nu}\Big|_{\partial\Omega} = 0 & t > 0, \\ u^\varepsilon(0,x) = u_0^\varepsilon(x) & x \in \Omega. \end{cases} \tag{AC}$$

$$E^\varepsilon[u^\varepsilon] := \int_\Omega \left(\frac{\varepsilon}{2} |\nabla u^\varepsilon|^2 + \frac{W(u^\varepsilon)}{\varepsilon} \right) dx =: \int_\Omega d\mu_t^\varepsilon.$$

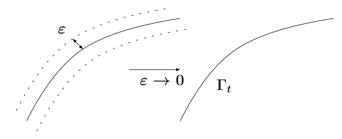
- Ilmanen (1993)

$$\mu_t^\varepsilon \rightharpoonup \mu_t \quad (\varepsilon \downarrow 0 \text{ sub seq.})$$

μ_t: weak mean curvature flow (Weak MCF)

Heuristic reason

- μ_t^ε is approximate area functional.
- (AC) is gradient flow of energy E^ε.



Weak mean curvature flow

$M_t = M_t^m \hookrightarrow \mathbb{R}^n$: smooth submanifolds for $t \geq 0$

(Classical) Mean curvature flow

$$V = H \quad \text{on } M_t, \quad t > 0$$

where V is a normal velocity vector, H is the mean curvature vector.

Derivation of the weak mean curvature flow

Manifolds $\longrightarrow$ Radon measures

For $\phi \in C_0^\infty(\mathbb{R}^n)$,

$$\begin{aligned}\frac{d}{dt}\int_{M_t} \phi \, d\mathscr{H}^m &= \int_{M_t} \nabla\phi \cdot V + \phi \operatorname{div}_{M_t} V \, d\mathscr{H}^m \\ &= \int_{M_t} (\nabla\phi - \phi H) \cdot H \, d\mathscr{H}^m.\end{aligned}$$

$\because \operatorname{div}_{M_t} V = |H| \operatorname{div}_{M_t} \vec{n} = -|H|^2 \quad$ ($\vec{n}$: normal vector on M_t)

Weak mean curvature flow

$$\frac{d}{dt}\int_{M_t}\phi\,d\mathscr{H}^m=\int_{M_t}(\nabla\phi-\phi H)\cdot H\,d\mathscr{H}^m$$

$\mu_t := \mathscr{H}^m\lfloor_{M_t}$

$$\Longrightarrow \frac{d}{dt}\int_{\mathbb{R}^n}\phi\,d\mu_t=\int_{\mathbb{R}^n}(\nabla\phi-\phi H)\cdot H\,d\mu_t$$

Definition (Weak mean curvature flow, Brakke (1978))

Family of Radon measures $\{\mu_t\}_{t\geq 0}$ on $\mathbb{R}^n$ satisfies weak MCF $\underset{\text{def.}}{\Leftrightarrow}$ for $\phi \in C_0^\infty(\mathbb{R}^n)$ satisfying $\phi \geq 0$,

$$\frac{d}{dt}\int_{\mathbb{R}^n}\phi\,d\mu_t\leq\int_{\mathbb{R}^n}(\nabla\phi-\phi H)\cdot H\,d\mu_t,$$

where H is generalized mean curvature vector.

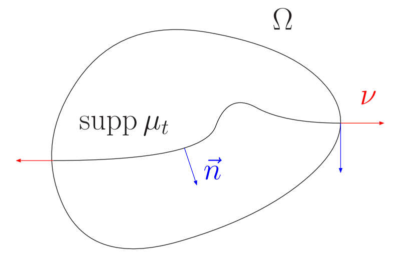
Problem and Motivation

<u>Problem</u>

Boundary behavior of μ_t ?

<u>Motivation</u>

1. How to study 2-codimensional submanifolds ?
2. Does μ_t satisfy (right-angle) boundary condition ?
   How to formulate the right-angle boundary condition ?
3. How does μ_t move on the boundary ?



Monotonicity formula

What is monotonicity formula ?

- Estimates about density ratio
- Invariant for blow-up arguments
- Plays the important role to study regularity theory

Known Results

- Allard (1975), Grüter-Jost (1985), Bourni (2012)
  boundary monotonicity formula and regularity theory for varifolds
- Y.-M. Chen-F.-H. Lin (1993)
  Boundary monotonicity formula for harmonic map heat flow
- Ilmanen (1993)
  Interior monotonicity formula for evolutional Allen-Cahn equation
- Tonegawa (2003)
  Boundary monotonicity formula for stationary Allen-Cahn equation

Boundary monotonicity formula

Theorem

$\Omega \subset \mathbb{R}^n$: *convex*, u_0^ε : *nice*

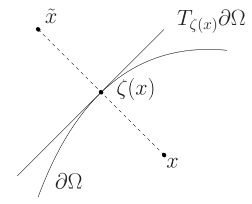
$\Longrightarrow \exists C > 0$ *s.t.* $\forall y \in \partial\Omega, \forall s > 0, 0 < \forall t < s$

$$\frac{d}{dt}\left(\exp\left(C(s-t)^{\frac{1}{4}}\right)\int_\Omega (\rho\eta + \tilde{\rho}\tilde{\eta})\, d\mu_t(x) + \textit{(Correction)}\right) \le 0$$

where, η, $\tilde{\eta}$ *is cut-off function,*

$$\rho(t,x) := \frac{1}{(4\pi(s-t))^{\frac{n-1}{2}}}\exp\left(-\frac{|x-y|^2}{4(s-t)}\right),$$

$$\tilde{\rho}(t,x) := \frac{1}{(4\pi(s-t))^{\frac{n-1}{2}}}\exp\left(-\frac{|\tilde{x}-y|^2}{4(s-t)}\right)$$



Remark $(\nabla\rho + \nabla\tilde{\rho})\cdot\nu\Big|_{\partial\Omega} \equiv 0$ (admissible condition)

More about monotonicity formula

$$\frac{d}{dt}\left(\exp\left(C(s-t)^{\frac{1}{4}}\right)\int_{\Omega}(\rho\eta+\tilde{\rho}\tilde{\eta})\,d\mu_t(x)+\text{(correction)}\right)\leq 0$$

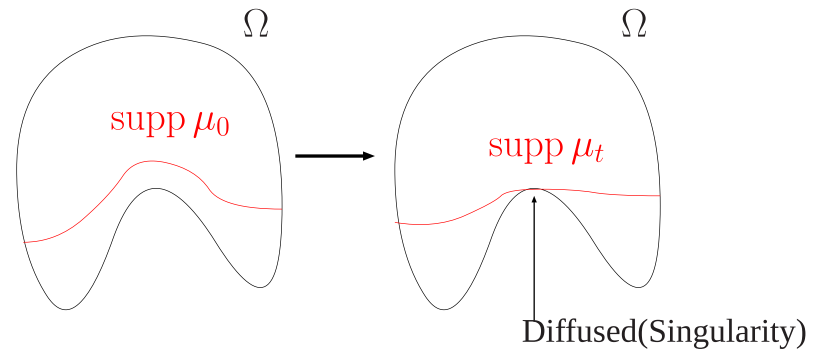
$$\begin{aligned}\int\rho\,d\mu_t(x)&=\frac{1}{(4\pi(s-t))^{\frac{n-1}{2}}}\int\exp\left(-\frac{|x-y|^2}{4(s-t)}\right)\,d\mu_t(x)\\&\geq\frac{1}{(\sqrt{2\pi}r)^{n-1}}\int_{B_r(y)}\exp\left(-\frac{|x-y|^2}{2r^2}\right)\,d\mu_t(x)\\&\geq c_n\frac{\mu_t(B_r(y))}{\mathscr{H}^{n-1}(B_r^{n-1}(y))}\end{aligned}$$

where $c_n>0$ is some constant and $r^2:=2(s-t)$, $t<s$.

$$\limsup_{r\downarrow 0}\frac{\mu_t(B_r(y))}{\mathscr{H}^{n-1}(B_r^{n-1}(y))}\leq\lim_{t\uparrow s}\frac{1}{c_n}\int\rho\,d\mu_t\leq\frac{1}{c_n}\int\rho(t_0,x)\,d\mu_{t_0}$$

Remark about convexity assumption

We may not remove the assumption about convexity for Ω



Diffused $\Longrightarrow$ Regularity is worse

$\Longrightarrow$ monotonicity formula may not hold

Key of proof –1st variation–

$\phi = \rho + \tilde{\rho}$ (For simplicity)

$$\frac{d}{dt}\int_\Omega \phi\, d\mu_t^\varepsilon \le \int_\Omega \left\{ \frac{(\nabla\phi\cdot\vec{n})^2}{\phi} - \mathrm{tr}\Big((\vec{n}\otimes\vec{n})D^2\phi\Big)\right\}\varepsilon|\nabla u|^2\,dx$$
$$- \int_\Omega \nabla\phi\cdot\nabla\left(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{W(u)}{\varepsilon}\right)\,dx + \int_\Omega \partial_t\phi\, d\mu_t^\varepsilon$$

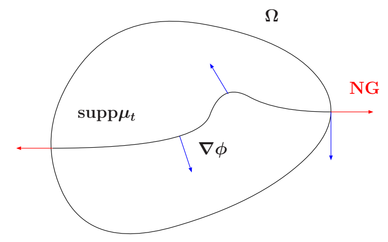
where $\vec{n} = \frac{\nabla u}{|\nabla u|}$ is approximate unit normal on μ_t

$$- \int_\Omega \nabla\phi\cdot\nabla\left(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{W(u)}{\varepsilon}\right)\,dx$$
$$= \int_\Omega \mathrm{div}(\nabla\phi)\, d\mu_t^\varepsilon - \int_{\partial\Omega}(\nabla\phi\cdot\nu)\left(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{W(u)}{\varepsilon}\right)\,d\sigma$$
$$\approx \delta\mu_t^\varepsilon[\nabla\phi] \qquad (\because \nabla\phi\cdot\nu = \nabla(\rho+\tilde{\rho})\cdot\nu \equiv 0)$$

Why we use reflection ?

$$
\begin{aligned}
&-\int_\Omega \nabla\phi\cdot\nabla\left(\frac{\varepsilon}{2}|\nabla u|^2+\frac{W(u)}{\varepsilon}\right)dx\\
&=\int_\Omega \mathrm{div}(\nabla\phi)\,d\mu_t^\varepsilon-\int_{\partial\Omega}(\nabla\phi\cdot\nu)\left(\frac{\varepsilon}{2}|\nabla u|^2+\frac{W(u)}{\varepsilon}\right)d\sigma\\
&\approx \delta\mu_t^\varepsilon[\nabla\phi] \qquad (\because \nabla\phi\cdot\nu=\nabla(\rho+\tilde{\rho})\cdot\nu\equiv 0)
\end{aligned}
$$

- $\nabla\phi$: variational vector field.
- $\nabla\phi\cdot\nu\Big|_{\partial\Omega}\not\equiv 0$ is not permitted



Further study

perpendicularity at the boundary

ϕ: test function satisfying $\nu \cdot \nabla \phi \Big|_{\partial\Omega} \equiv 0$

$$\int_\Omega \operatorname{div} \nabla \phi \, d\mu_t = - \int_\Omega \nabla \phi \cdot H \, d\mu_t \quad \text{(weak perpendicularity)}$$

Does we obtain classical perpendicularity at the boundary ?

partial regularity up to the boundary

- Brakke (1978), Kasai-Tonegawa (2012)
  $\{\mu_t\}_{t>0}$: k-dim. weak MCF, S_t: Singular set
  $\Rightarrow \mathscr{H}^k(S_t) = 0$ for almost every $t > 0$
  $(\dim_{\mathscr{H}}(S_t) \le k)$

Does we extend this results up to the boundary ?

$\{\mu_t\}_{t>0}$ $(n-1)$-dim $\Rightarrow \dim_{\mathscr{H}}(\partial\Omega \cap \operatorname{spt} \mu_t) \approx n-2$

$\partial\Omega \cap S_t$ is null set ? Hausdorff dimension ?