Boundary monotonicity formula for the evolutional Allen-Cahn equation with Neumann boundary condition

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Allen-Cahn equation

$$\begin{cases} \varepsilon u_t^\varepsilon = \varepsilon \Delta u^\varepsilon - \dfrac{W'(u^\varepsilon)}{\varepsilon} & t > 0,\ x \in \Omega, \\ \quad = -\delta E^\varepsilon[u^\varepsilon] & \\ \dfrac{\partial u^\varepsilon}{\partial \nu}\Big|_{\partial\Omega} = 0 & t > 0, \\ u^\varepsilon(0, x) = u_0^\varepsilon & x \in \Omega. \end{cases} \qquad \text{(AC)}$$

Ω: bounded domain, $\partial\Omega$: smooth enough,

ν: outer unit normal to $\partial\Omega$

$\varepsilon > 0$, $W(u^\varepsilon) := (1 - (u^\varepsilon)^2)^2$

$$E^\varepsilon[u^\varepsilon] := \int_\Omega \left(\frac{\varepsilon}{2} |\nabla u^\varepsilon|^2 + \frac{W(u^\varepsilon)}{\varepsilon} \right) dx =: \int_\Omega d\mu_t^\varepsilon.$$

- Ilmanen (1993)

$$\mu_t^\varepsilon \rightharpoonup \mu_t \quad (\varepsilon \downarrow 0)$$

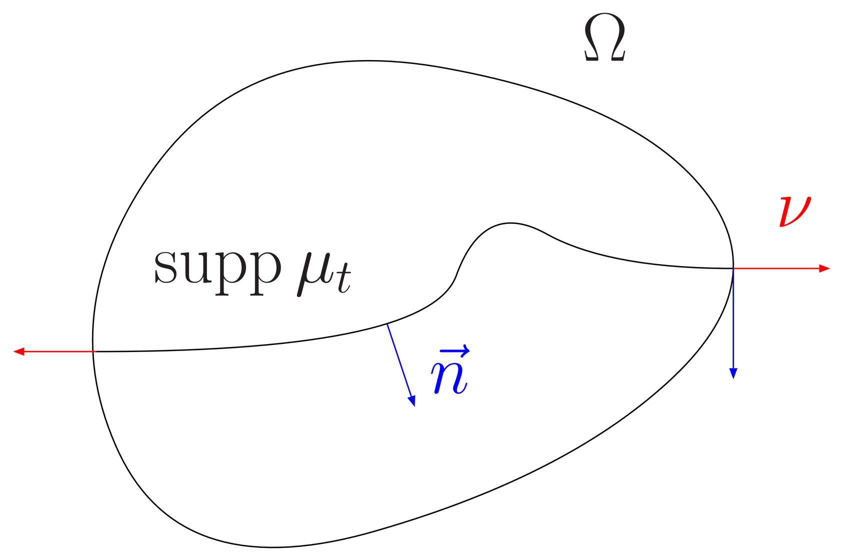
μ_t: (loosely speaking) Brakke's mean curvature flow

Problem

To study the boundary behavior of μ_t.

Motivation

1. How to study 2 codimensional submanifolds ?
2. Is μ_t Brakke' mean curvature flow up to the boundary ?
3. How μ_t moves at the boundary ?



Our main study

We derive the boundary monotonicity formula.

Known results

- Interior monotonicity formula is well-known.
  - Schoen (Harmonic map)
  - Huisken (Mean curvature flow)
  - Struwe (Harmonic map heat flow)
  - Giga-Kohn (nonlinear heat equation)
- Ilmanen (1993)
  derive Huisken's type interior monotonicity formula for the Allen-Cahn equation
- Allard (1975), Grüter-Jost (1985), Bourni (2012)
  derive the boundary monotonicity formula for stationary varifolds
- Y.-M. Chen-F.-H. Lin (1993)
  derive the boundary monotonicity formula for harmonic map heat flow with Dirichlet boundary condition
- Tonegawa (2003)
  derive the boundary monotonicity formula for the stationary Allen-Cahn equation with Neumann boundary condition

We show for the evolutional Allen-Cahn equation with Neumann boundary condition

Notation

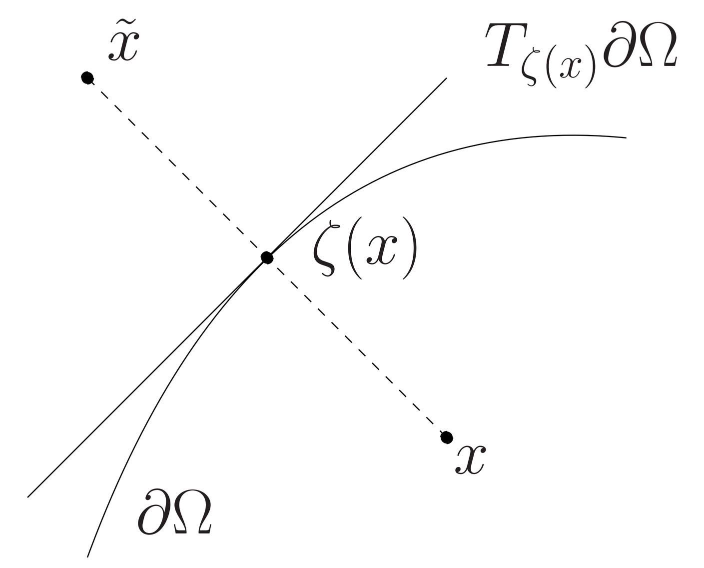
$x \in \Omega \cup \partial\Omega$: sufficiently near $\partial\Omega$
(belonging to the tubular neighborhood of $\partial\Omega$)

$\zeta(x) \in \partial\Omega$: minimizer of $\mathrm{dist}(x, \partial\Omega)$
$(|x - \zeta(x)| = \mathrm{dist}(x, \partial\Omega))$

$\tilde{x} = 2\zeta(x) - x$: reflection point of x with respect to $T_{\zeta(x)}\partial\Omega$

$$\tilde{x} = 2\zeta(x) - x = \zeta(x) + (\zeta(x) - x)$$

Figure of $\zeta(x)$ and $\tilde{x}$



Backward heat kernel

$y \in \partial\Omega$, $0 < t < s$, $x \in \Omega$, $x \in \Omega$: near $\partial\Omega$

$$\rho(t, x) := \frac{1}{(4\pi(s-t))^{\frac{n-1}{2}}} \exp\left(-\frac{|x-y|^2}{4(s-t)} \right)$$

$$\tilde{\rho}(t, x) := \frac{1}{(4\pi(s-t))^{\frac{n-1}{2}}} \exp\left(-\frac{|\tilde{x}-y|^2}{4(s-t)} \right)$$

$\eta = \eta(r)$: cut-off function satisfying $0 \le \eta \le \chi_{[0,R]}$ for some $R > 0$

Theorem (Tonegawa-M.)

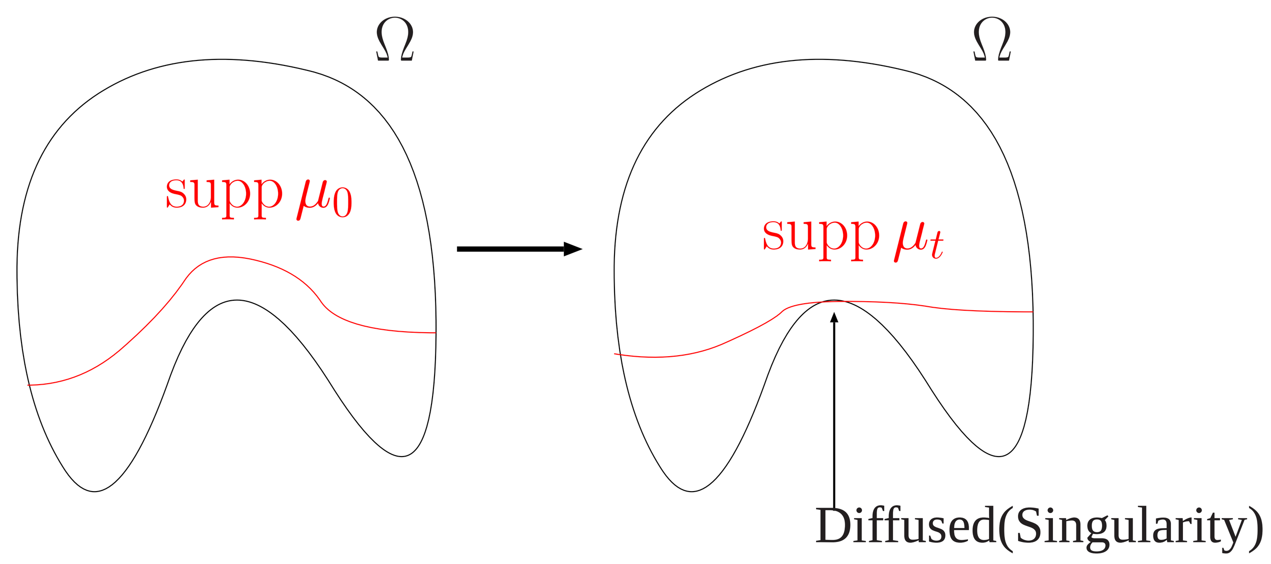
$\forall y \in \Omega, \forall s > 0, \exists C_1, C_2 > 0$ *(depending only on $n, M, \partial\Omega$) s.t.*

$$\frac{d}{dt}\left(\exp\left(C_1 (s-t)^{\frac{1}{4}}\right) \int_\Omega (\rho\eta(|x-y|) + \tilde{\rho}\eta(|\tilde{x}-y|))\, d\mu_t + C_2 \int_t^s \exp\left(C_1(s-\tau)^{\frac{1}{4}}\right) \left(1 + \frac{1}{\sqrt{s-\tau}}\right) d\tau \right) \le 0$$

$$(0 < \forall t < s)$$

Remark

It seems that convexity assumption can not be removed.



Why we consider the reflection ?

$\phi = \phi(t, x) \ge 0$: nice function

$$d\mu_t^\varepsilon := \left(\frac{\varepsilon}{2}|\nabla u^\varepsilon|^2 + \frac{\tilde{W}(u^\varepsilon)}{\varepsilon} \right) dx, \quad d\xi_t^\varepsilon := \left(\frac{\varepsilon}{2}|\nabla u^\varepsilon|^2 - \frac{\tilde{W}(u^\varepsilon)}{\varepsilon} \right) dx$$

Then we obtain

$$\begin{aligned} \frac{d}{dt}\int_\Omega \phi\, d\mu_t^\varepsilon \le & \int_\Omega \left(\frac{(p \cdot \nabla\phi)^2}{\phi} + (I - p \otimes p, D^2\phi)_{M_n(\mathbb{R})} \right) d\mu_t^\varepsilon \\ & + \int_\Omega \left(\frac{(p \cdot \nabla\phi)^2}{\phi} - (p \otimes p, D^2\phi)_{M_n(\mathbb{R})} \right) d\xi_t^\varepsilon \\ & + \int_\Omega \phi_t\, d\mu_t^\varepsilon - \int_{\partial\Omega} \left(\frac{|\nabla u^\varepsilon|^2}{2} + \frac{W(u^\varepsilon)}{\varepsilon} \right) (\nabla\phi \cdot \nu)\, d\sigma. \end{aligned}$$

$$\phi = \rho + \tilde{\rho} \Longrightarrow \nabla\phi \cdot \nu\Big|_{\partial\Omega} \equiv 0$$

Remark

From the point of differential geometry, this is the first variation and the variational vector field is restricted to the tangential direction on $\partial\Omega$.

Further study

- Vanishing of the discrepancy ($d\xi_t^\varepsilon dt \rightharpoonup 0$)
- Existence of Brake's mean curvature flow up to the boundary
- Construct the boundary measure and study its behavior
- Other boundary condition (e.g. dynamical boundary condition)