Hölder regularity for solutions of the evolutional p-Laplace equation with external forces

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evolutional p-Laplace equation

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$$\begin{cases} \partial_t u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \operatorname{div} f, & t > 0, x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{p-P}$$

$f = (f_1(t,x), \dots, f_n(t,x))$: given, $p > 2$.

$-\operatorname{div}(|\nabla u|^{p-2}\nabla u)$: p-Laplace operator (1st variation of $\frac{1}{p}\|\nabla u\|_p^p$)

$$\because \quad \frac{d}{d\varepsilon}\|\nabla(u+\varepsilon\phi)\|_p^p\Big|_{\varepsilon=0} = p\int |\nabla u|^{p-2}\nabla u \cdot \nabla\phi \, dx = -p\int \operatorname{div}(|\nabla u|^{p-2}\nabla u)\phi \, dx$$

- $p = 2$: uniformly parabolic (heat equations)
- $p > 2$: degenerate parabolic ($|\nabla u| = 0 \Rightarrow |\nabla u|^{p-2} = 0$)
- $p < 2$: singular parabolic ($|\nabla u| = 0 \Rightarrow |\nabla u|^{p-2} = \infty$)

Barenblatt solution

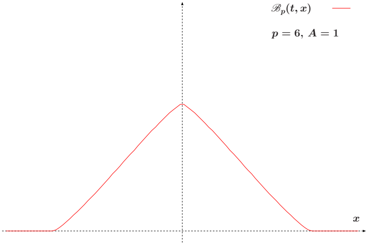
$$\partial_t u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \operatorname{div} f, \quad t > 0,\ x \in \mathbb{R}^n, \qquad (p\text{-P})$$

The Barenblatt solution $\quad f \equiv 0$

$$\mathscr{B}_p(t,x) = \frac{1}{(1+\sigma t)^{\frac{n}{\sigma}}} \left(A - \frac{p-2}{p} \left(\frac{|x|}{(1+\sigma t)^{\frac{1}{\sigma}}} \right)^{\frac{p}{p-1}} \right)_+^{\frac{p-1}{p-2}}$$

where $\sigma = n(p-2)+p$, $A > 0$.

- not twice differentiable
- self-similar solution



weak solution

evolutional p-Laplace equation

$$\begin{cases} \partial_t u - \mathrm{div}(|\nabla u|^{p-2}\nabla u) = \mathrm{div}\, f, & t > 0, x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \qquad (p\text{-P})$$

Definition (weak solution)

u: weak solution of (p-P) $\underset{\text{def}}{\Longleftrightarrow}$ $T > 0$ s.t.

- $u \in L^\infty(0,T\,;\, L^2(\mathbb{R}^n))$ and $\nabla u \in L^p((0,T) \times \mathbb{R}^n)$
- For $\phi \in C^1(0,T; C_0^1(\mathbb{R}^n))$ and for almost all $0 < t < T$,

$$\int_{\mathbb{R}^n} u(t)\phi(t)\,dx + \int_0^t \int_{\mathbb{R}^n} (-u\partial_t\phi + |\nabla u|^{p-2}(\nabla u \cdot \nabla\phi))\,dxd\tau = \int_{\mathbb{R}^n} u_0\phi(0)\,dx - \int_0^t \int_{\mathbb{R}^n} (f \cdot \nabla\phi)\,dxd\tau.$$

Problem and Motivation

$$\partial_t u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \operatorname{div} f, \quad t > 0,\ x \in \mathbb{R}^n, \qquad (p\text{-P})$$

Problem

Derive Hölder continuity of ∇u, namely

$$|\nabla u(t,x) - \nabla u(s,y)| \le C(|t-s|^{\frac{\gamma}{2}} + |x-y|^{\gamma}).$$

Motivation $\quad n = 1,\ v := \partial_x u$

$$\Longrightarrow \partial_t v - \partial_x^2(|v|^{p-2}v) = \partial_x(\partial_x f). \qquad \text{(porous medium equation)}$$

$$\frac{2}{r} + \frac{n}{q} < 1, \quad \|\partial_x f\|_{L_t^r L_x^q} := \left(\int \left(\int |\partial_x f|^q \, dx\right)^{\frac{r}{q}} dt\right)^{\frac{1}{q}} < \infty$$

$\Longrightarrow v$: Hölder continuous

Known Results and Main Theorem

$$\partial_t u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \operatorname{div} f, \quad t > 0,\ x \in \mathbb{R}^n. \qquad (p\text{-P})$$

Known Results

- DiBenedetto-Friedman ('85), Wiegner ('86)

$$f \equiv 0 \Longrightarrow \nabla u : \text{Hölder continuous}$$

- Misawa ('02)

f : Hölder continuous with respect to (t, x)

$\Longrightarrow \nabla u$: Hölder continuous

Theorem

$\frac{2}{r} + \frac{n}{q} < 1$, $\nabla f \in L^r(0, \infty; L^q(\mathbb{R}^n)) \Longrightarrow \nabla u$: *Hölder continuous.*

Remark for Main Theorem

$$\partial_t u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \operatorname{div} f, \quad t > 0,\ x \in \mathbb{R}^n. \qquad (p\text{-P})$$

Theorem (Before mentioned)

$\frac{2}{r} + \frac{n}{q} < 1$, $\nabla f \in L^r(0,\infty\,;\,L^q(\mathbb{R}^n)) \Longrightarrow \nabla u$: *Hölder continuous.*

Let $\frac{2}{r} + \frac{n}{q} < 1$, $\nabla f \in L^r_t L^q_x$. Then

- f is Hölder continuous to x ($\because q > n$ and Sobolev inequality)
- f needs not to be continuous to t.

example for the external force

$f(t,x) = f_0(t)x$ for some $f_0 \in L^r(0,\infty)$, $r > 2$

conclusion

When f is singular with respect to t, we may cancel the singularity by the smoothness with respect to x.

scaling argument

$$\partial_t u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \operatorname{div} f, \quad t > 0,\ x \in \mathbb{R}^n. \qquad (p\text{-P})$$

Theorem (Before mentioned)

$\frac{2}{r} + \frac{n}{q} < 1$, $\nabla f \in L^r(0,\infty\,;\,L^q(\mathbb{R}^n)) \Longrightarrow \nabla u$: *Hölder continuous.*

$0 < \rho \ll 1$, $t = \rho^2 s$, $y = \rho x$, $(\nabla_y = \rho \nabla_x)$

$u_\rho(s,y) = \dfrac{1}{\rho} u(t,x)$, $f_\rho(s,y) = f(t,x)$

$$\Longrightarrow \partial_s u_\rho - \operatorname{div}_y(|\nabla_y u_\rho|^{p-2}\nabla_y u_\rho) = \operatorname{div}_y f_\rho, \quad \nabla_x u = \nabla_y u_\rho$$

$$\|\nabla_y f_\rho\|_{L^r_s L^q_y} = \rho^{1-\frac{2}{r}-\frac{n}{q}} \|\nabla_x f\|_{L^r_t L^q_x}$$

$$1 - \frac{2}{r} - \frac{n}{q} > 0 \Leftrightarrow \frac{2}{r} + \frac{n}{q} < 1 \Rightarrow \text{perturbation is small}$$

Integral condition of Hölder continuity(Campanato condition)

Notation

$B_R := \{x \in \mathbb{R}^n : |x| < R\}, Q_R := (-R^2, 0) \times B_R$

The Campanato condition

$$|\nabla u(t,x) - \nabla u(s,y)| \leq C(|t-s|^{\frac{\gamma}{2}} + |x-y|^{\gamma})$$

$$\underset{\text{equiv.}}{\Longleftrightarrow} \iint_{Q_R} |\nabla u - (\nabla u)_{Q_R}|^p \, dxdt \leq C R^{n+2+p\gamma}$$

where $(\nabla u)_{Q_R} = \dfrac{1}{|Q_R|} \iint_{Q_R} \nabla u \, dxdt$.

$$\begin{aligned} \because & \iint_{Q_R} |\nabla u - (\nabla u)_{Q_R}|^p \, dxdt \\ \leq & \iint_{Q_R} \left| \frac{1}{|Q_R|} \iint_{Q_R} |\nabla u(t,x) - \nabla u(s,y)| \, dsdy \right|^p \, dxdt \\ \leq & \, C R^{n+2+p\gamma}. \end{aligned}$$

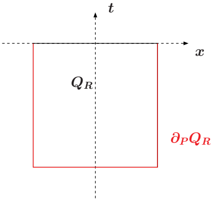
Reference equation

$$\partial_t u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \operatorname{div} f, \quad Q_R. \qquad (p\text{-P})$$

Reference equation

$$\begin{cases} \partial_t v - \operatorname{div}(|\nabla v|^{p-2}\nabla v) = 0, & \text{in } Q_R, \\ v = u, & \text{on } \partial_P Q_R \end{cases} \qquad \text{(RE)}$$

$$\partial_P Q_R = \{-R^2\} \times B_R \cup (-R^2, 0) \times \partial B_R$$



Lemma

$\exists C = C(n,p) > 0$ s.t.

$$\iint_{Q_R} |\nabla u - \nabla v|^p \, dxdt \le C \|\nabla f\|_{L^r_t L^q_x}^{\frac{p}{p-1}} R^{n+2+\frac{p}{p-1}(1-\frac{n}{q}-\frac{2}{r})}.$$

proof of Lemma

$$\partial_t u - \mathrm{div}(|\nabla u|^{p-2}\nabla u) = \mathrm{div}\, f = \mathrm{div}(f - (f(t))_{B_R}) \quad \text{in } Q_R. \qquad (p\text{-P})$$

$$\partial_t v - \mathrm{div}(|\nabla v|^{p-2}\nabla v) = 0 \quad \text{in } Q_R, \qquad \text{(RE)}$$

$$v = u \quad \text{on } \partial_P Q_R$$

Consider $\iint_{Q_R} ((p\text{-P}) - \text{(RE)})(u - v)\, dxdt$

$$C_0 \iint |\nabla u - \nabla v|^p \, dxdt$$

$$\leq \iint (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v) \cdot (\nabla u - \nabla v)\, dxdt$$

$$\leq - \iint (f - (f(t))_{B_R}) \cdot (\nabla u - \nabla v)\, dxdt$$

$$\leq \frac{C_0}{2} \iint |\nabla u - \nabla v|^p \, dxdt + C \iint |f - (f(t))_{B_R}|^{\frac{p}{p-1}} \, dxdt$$

proof of Theorem

By the Poincare inequality and the Young inequality,

$$\iint |\nabla u - \nabla v|^p \, dxdt \le C \iint |f - (f(t))_{B_R}|^{\frac{p}{p-1}} \, dxdt$$
$$\le C \|\nabla f\|_{L_t^r L_x^q}^{\frac{p}{p-1}} R^{n+2+\frac{p}{p-1}(1-\frac{n}{q}-\frac{2}{r})}.$$

$$\therefore \iint_{Q_R} |\nabla u - (\nabla u)_{Q_R}|^p \, dxdt \le C \Big(\iint_{Q_R} |\nabla u - \nabla v|^p \, dxdt$$
$$+ \iint_{Q_R} |\nabla v - (\nabla v)_{Q_R}|^p \, dxdt$$
$$+ \iint_{Q_R} |(\nabla v)_{Q_R} - (\nabla u)_{Q_R}|^p \, dxdt \Big) \le C R^{n+2+p\gamma}$$

for some $\gamma > 0 \Longrightarrow \nabla u$ is Hölder continuous