REMARKS ON HÖLDER CONTINUITY FOR SOLUTIONS OF THE p-LAPLACE EVOLUTION EQUATIONS

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ABSTRACT. We study the interior Hölder regularity problem for the gradient of solutions of the p-Laplace evolution equations with the external forces. Misawa give some conditions for the Hölder continuity of the gradient of solutions. We show Hölder estimates of the solutions with weaker condition as for Misawa.

1. INTRODUCTION

We consider the following p-Laplace evolution equations:

$$\begin{cases} \partial_t u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \operatorname{div} f, & t > 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $p > \frac{2n}{n+2}$ is a constant, $u : (0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ is unknown, $f : (0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ and $u_0 : \mathbb{R}^n \to \mathbb{R}$ are given external and initial data. It is well-known that a classical solution of (1.1) does not generally exist. Hence we introduce the notion of weak solutions.

Definition 1.1. For $u_0 \in L^2(\mathbb{R}^n)$ and $f \in L^{\frac{p}{p-1}}((0,\infty) \times \mathbb{R}^n)$, we call u a weak solution of (1.1) if there exists $T > 0$ such that

(1) $u \in L^\infty(0,T\,; L^2(\mathbb{R}^n))$ with $\nabla u \in L^p(0,T\,; L^p(\mathbb{R}^n))$; and
(2) u satisfies (1.1) in the sense of distribution, namely, for all $\varphi \in C^1(0,T\,; C_0^1(\mathbb{R}^n))$ and for almost all $0 < t < T$,

$$\int_{\mathbb{R}^n} u(t)\varphi(t)\,dx - \int_0^t \int_{\mathbb{R}^n} u\partial_t\varphi\,d\tau dx + \int_0^t \int_{\mathbb{R}^n} |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi\,d\tau dx = \int_{\mathbb{R}^n} u_0\varphi(0)\,dx - \int_0^t \int_{\mathbb{R}^n} f \cdot \nabla\varphi\,d\tau dx.$$

For the existence of a weak solution is shown by Browder [1], Ladyženskaja-Solonnikov-Ural'ceva [7] (cf. Ôtani [10]). In this paper, we study the Hölder continuity of ∇u, particularly, we show a relationship between the Hölder continuity of ∇u and the regularity of the external force f.

The Hölder continuity of ∇u was firstly shown by DiBenedetto-Friedman [4, 5] and Wiegner [11] when $f \equiv 0$. Misawa [8] showed the Hölder continuity of ∇u if f is locally Hölder continuous with respect to t and x. In this paper, we give a weaker condition of the external force f for the Hölder continuity of ∇u than the condition given by Misawa.

To state the main theorem, we introduce some notation. For $t_0 \in \mathbb{R}$ and $R > 0$, we write an open interval $I_R(t_0) := (t_0 - R^2, t_0)$. For $x_0 \in \mathbb{R}^n$ and $R > 0$, we denote the n-dimensional open ball with radius R and center x_0 by $B_R(x_0)$. We define a parabolic cylinder $Q_R(t_0, x_0)$ by $Q_R(t_0, x_0) := I_R(t_0) \times B_R(t_0)$. For an integrable function f on a measurable set A, we denote an integral mean $(f)_A$ by

$$(f)_A := \frac{1}{|A|}\int_A f\,dx.$$

Theorem 1.2. *Assume that for some constant $K > 0$ and $\gamma > n+2$, the external force f satisfies*

$$\iint_{Q_R(t_0,x_0)} |f - (f(t))_{B_R(x_0)}|^{\frac{p}{p-1}}\,dtdx \le KR^\gamma \tag{1.2}$$

for all $(t_0,x_0) \in (0,\infty)\times\mathbb{R}^n$ and $0 < R < 1$ satisfying $Q_R(t_0,x_0) \subset (0,\infty)\times\mathbb{R}^n$. Let u be a weak solution of (1.1) *Then ∇u is Hölder continuous with exponent $\gamma > 0$ depending only on n, p, γ. Furthermore, for all $\varepsilon > 0$, there exists a constant $C > 0$ depending only on $n, p, \gamma, K, \varepsilon$ and $\|\nabla u\|_{L^\infty((0,\infty)\times\mathbb{R}^n)}$ such that*

$$|\nabla u(t,x) - \nabla u(s,y)| \le C(|t-s|^{\frac{\gamma}{2}} + |x-y|^\gamma).$$

for all $(t,x), (s,y) \in (\varepsilon,\infty)\times\mathbb{R}^n$.

Remark 1.3. If f is Hölder continuous in $(0,\infty)\times\mathbb{R}^n$, then the assumption (1.2) holds. Furthermore, if $\nabla f \in L^r(0,\infty\,;\,L^q(\mathbb{R}^n))$ with $\frac{2}{r}+\frac{n}{q} < 1$, then the assumption (1.2) holds. Thus, we need not the assumption that f is Hölder continuous with respect to t.

Remark 1.4. The assumption that ∇u belongs to $L^\infty((0,\infty)\times\mathbb{R}^n)$ can be replaced by the assumption that for some $M > 0$,

$$\iint_{Q_R(t_0,x_0)} |\nabla u|^p\,dxdt \le MR^{n+2-\sigma} \tag{1.3}$$

for sufficiently small $\sigma > 1$ and for $Q_R(t_0,x_0) \subset (0,\infty)\times\mathbb{R}^n$, namely Morrey regularity for $|\nabla u|^p$. If the external force f is in the space of functions of bounded mean oscillation, then $\nabla u \in L^r((0,\infty)\times\mathbb{R}^n)$ for all $r > p$ by Misawa [9] and the assumption (1.3) holds. Since the assumption (1.2) implies the boundedness of mean oscillation of f, the boundedness of ∇u is not an essential assumption.

Remark 1.5. We only treat the case $p > 2$ in this paper since we may obtain the Hölder continuity of ∇u for the case $\frac{2n}{n+2} < p < 2$ by almost same argument. Especially, we may show the Morrey regularity of $|\nabla u|^p$ for the case $\frac{2n}{n+2} < p$. Furthermore, even for the case $1 < p \le \frac{2n}{n+2}$, we may also show the local Hölder continuity of ∇u by assuming the Morrey regularity of $|\nabla u|^p$ (cf. Choe [2]).

At the end of this section, we introduce further notation. For a parabolic cylinder $Q_R(t_0,x_0)$, we define a parabolic boundary $\partial_p Q_R(t_0,x_0)$ by

$$\partial_p Q_R(t_0,x_0) := (\{t_0 - R^2\}\times B_R(x_0)) \cup (I_R(t_0)\times\partial B_R(x_0)).$$

For $(t,x),(s,y) \in \mathbb{R}\times\mathbb{R}^n$, we write a parabolic distance $\mathrm{dist}_p((t,x),(s,y))$ by

$$\mathrm{dist}_p((t,x),(s,y)) := \max\{|t-s|^{\frac{1}{2}}, |x-y|\}.$$

For $A, B \subset \mathbb{R}\times\mathbb{R}^n$, we write a parabolic distance $\mathrm{dist}_p(A,B)$ by

$$\mathrm{dist}_p(A,B) := \inf_{z\in A, z'\in B} \mathrm{dist}_p(z,z').$$

We denote a constant depending on $\alpha, \beta, \dots$ by $C(\alpha,\beta,\dots)$. The same letter C will be used to denote difference constants. We use subscript numbers if we consider the relation between the constants.

2. PROOF OF THEOREM 1.2

From Remark 1.4 and Remark 1.5, we may assume without loss of generality that $p > 2$ and the boundedness of the gradient of the solution. By the scaling argument, we only consider $(t_0, x_0) \in (1, \infty) \times \mathbb{R}^n$ and we will show the Hölder continuity at (t_0, x_0). We abbreviate the center of open balls and parabolic cylinders.

For $R < 1$, we consider the following reference equation:

$$\begin{cases} \partial_t v - \operatorname{div}(|\nabla v|^{p-2}\nabla v) = 0, & (t,x) \in Q_R, \\ v = u, & (t,x) \in \partial_p Q_R. \end{cases} \tag{2.1}$$

For the existence of a solution of (1.2), we refer to Ladyženskaja-Solonnikov-Ural'ceva [7, Theorem 6.7 in p.466]

Lemma 2.1. *There exists a constant $C > 0$ depending only on n, p such that*

$$\int_{B_R} (v-u)^2 \, dx \Big|_{t=0} + \iint_{Q_R} |\nabla v - \nabla u|^p \, dtdx \le C \iint_{Q_R} |f - (f(t))_{B_R}|^{\frac{p}{p-1}} \, dtdx.$$

Proof. Subtract (1.1) from (2.1), multiply $(v-u)$ and integrate in Q_R. Then we obtain

$$\frac{1}{2}\iint_{Q_R} \partial_t (v-u)^2 \, dtdx + \iint_{Q_R} (|\nabla v|^{p-2}\nabla v - |\nabla u|^{p-2}\nabla u)\cdot(\nabla v - \nabla u)\, dtdx$$
$$= \iint_{Q_R} (f - (f(t))_{B_R})\cdot(\nabla v - \nabla u)\, dtdx.$$

We utilize algebraic inequalities

$$(|\mathbf{p}|^{p-2}\mathbf{p} - |\mathbf{q}|^{p-2}\mathbf{q})\cdot(\mathbf{p}-\mathbf{q}) \ge C_0|\mathbf{p}-\mathbf{q}|^p,$$

which hold for all $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$, where C_0 is a positive constant depending only on n, p (cf. DiBenedetto [3, Lemma 4.4 in p.13]). By the Hölder inequality and the Young inequality, we have

$$\begin{aligned}
&\frac{1}{2}\iint_{Q_R} \partial_t (v-u)^2 \, dtdx + C_0(n,p)\iint_{Q_R} |\nabla v - \nabla u|^p \, dtdx \\
&\le \iint_{Q_R} (f - (f(t))_{B_R})\cdot(\nabla v - \nabla u)\, dtdx \\
&\le \left(\iint_{Q_R} |f - (f(t))_{B_R})|^{\frac{p}{p-1}} \, dtdx\right)^{1-\frac{1}{p}} \left(\iint_{Q_R} |\nabla v - \nabla u|^p \, dtdx\right)^{\frac{1}{p}} \\
&\le \frac{C_0(n,p)}{2}\iint_{Q_R} |\nabla v - \nabla u|^p \, dtdx + C_1(n,p)\iint_{Q_R} |f - (f(t))_{B_R})|^{\frac{p}{p-1}} \, dtdx.
\end{aligned}$$

□

Remark 2.2. When $1 < p < 2$, we utilize the following algebraic inequalities:

$$(|\mathbf{p}|^{p-2}\mathbf{p} - |\mathbf{q}|^{p-2}\mathbf{q})\cdot(\mathbf{p}-\mathbf{q}) \ge C_0|\mathbf{p}-\mathbf{q}|^2(|\mathbf{p}|+|\mathbf{q}|),$$

which hold for all $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$, where C_0 is a positive constant depending only on n, p.

The following lemma is given by DiBenedetto [3].

Lemma 2.3 (DiBenedetto [3, Theorem 5.1 in p.238])**.** *There exists a constant $C > 0$ depending only on n, p such that*

$$\sup_{Q_{\frac{R}{2}}} |\nabla v| \le C\bigg(\frac{1}{|Q_R|}\iint_{Q_R} |\nabla v|^p \, dtdx\bigg)^{\frac{1}{2}}.$$

Using Lemma 2.3, we obtain the following lemma:

Lemma 2.4. *There exists a constant $C > 0$ depending only on n, p such that*

$$\sup_{Q_{\frac{R}{2}}} |\nabla v| \le C\bigg\{K^{\frac{1}{2}} R^{\frac{1}{2}(\gamma-n-2)} + \|\nabla u\|^{\frac{p}{2}}_{L^\infty((0,\infty)\times\mathbb{R}^n)}\bigg\}. \tag{2.2}$$

In particular, $|\nabla v|$ is bounded on $Q_{\frac{R}{2}}$.

Proof. By Lemma 2.1 and the assumption (1.2), we have

$$\begin{aligned}\iint_{Q_R} |\nabla v|^p \, dtdx &\le C(p)\bigg(\iint_{Q_R} |\nabla v - \nabla u|^p \, dtdx + \iint_{Q_R} |\nabla u|^p \, dtdx\bigg)\\ &\le C(p)\bigg(\iint_{Q_R} |f - (f(t))_{B_R}|^{\frac{p}{p-1}} \, dtdx + |Q_R| \|\nabla u\|^p_{L^\infty((0,\infty)\times\mathbb{R}^n)}\bigg)\\ &\le C(p)\bigg(KR^\gamma + |Q_R| \|\nabla u\|^p_{L^\infty((0,\infty)\times\mathbb{R}^n)}\bigg).\end{aligned}$$

Using Lemma 2.3, we obtain (2.2). □

Remark 2.5. If the assumption (1.3) holds, then we have the following growth estimate of $|\nabla v|$:

$$\sup_{Q_{\frac{R}{2}}} |\nabla v| \le C(K^{\frac{1}{2}} + M^{\frac{1}{2}})R^{-\frac{\sigma}{2}}.$$

The next lemma is given by DiBenedetto [3].

Lemma 2.6 (DiBenedetto [3, Theorem 1.1' in p.256])**.** *For $0 < \delta < 1$ and $R < 1$, there exist constants $\alpha_0 > 0$ and $C > 0$ depending only on n, p such that*

$$\operatorname*{osc}_{Q_{\frac{R}{2}}}(\nabla v) \le C\|\nabla v\|_{L^\infty(Q_{\frac{R^{1-\delta}}{2}})} \left(\frac{R + R\max\{1, \|\nabla v\|^{\frac{p-2}{2}}_{L^\infty(Q_{\frac{R^{1-\delta}}{2}})}\}}{\operatorname{dist}_p(Q_{\frac{R}{2}}, \partial_p Q_{\frac{R^{1-\delta}}{2}})}\right)^{\alpha_0}.$$

Using Lemma 2.6 we will show the following lemma:

Lemma 2.7. *For $0 < \delta < 1$, there exists a constant $C > 0$ depending only on n, p such that,*

$$\operatorname*{osc}_{Q_{\frac{R}{2}}}(\nabla v) \le 8R^\delta \|\nabla v\|_{L^\infty(Q_{\frac{R^{1-\delta}}{2}})} + C\|\nabla v\|_{L^\infty(Q_{\frac{R^{1-\delta}}{2}})}\Big(1 + \max\{1, \|\nabla v\|^{\frac{p-2}{2}}_{L^\infty(Q_{\frac{R^{1-\delta}}{2}})}\}\Big)^{\alpha_0} R^{\delta\alpha_0}.$$

Proof. Either if $R^\delta \ge \frac{1}{4}$, then

$$\operatorname*{osc}_{Q_{\frac{R}{2}}}(\nabla v) \le 2\|\nabla v\|_{L^\infty(Q_{\frac{R}{2}})} \le 8R^\delta \|\nabla v\|_{L^\infty(Q_{\frac{R^{1-\delta}}{2}})}.$$

Otherwise, if $0 < R^\delta < \frac{1}{4}$, then

$$\begin{aligned}\mathrm{dist}(Q_{\frac{R}{2}}, \partial_p Q_{\frac{R^{1-\delta}}{2}}) &= \min\left\{ \left(\left(\frac{R^{1-\delta}}{2} \right)^2 - \left(\frac{R}{2} \right)^2 \right)^{\frac{1}{2}}, \left(\frac{R^{1-\delta}}{2} - \frac{R}{2} \right) \right\} \\ &= \frac{R^{1-\delta}}{2}(1 - R^\delta) \geq \frac{3}{8} R^{1-\delta}.\end{aligned}$$

Therefore, by Lemma 2.6, we obtain

$$\operatorname*{osc}_{Q_{\frac{R}{2}}}(\nabla v) \leq C(p,n) \|\nabla v\|_{L^\infty(Q_{\frac{R^{1-\delta}}{2}})} \Big(1 + \max\{1, \|\nabla v\|^{\frac{p-2}{2}}_{L^\infty(Q_{\frac{R^{1-\delta}}{2}})}\} \Big)^{\alpha_0} R^{\delta \alpha_0}.$$

□

We slightly modify an iteration argument, since the monotonicity of the oscillation of ∇u is not easily obtained.

Lemma 2.8 (cf. Giaquinta [6, Lemma 2.1 in p.86]). *Let $\phi = \phi(s)$ be a non-negative function on $[0, \infty)$. Assume that for some constants $R_0, A_0, A_1, B, \alpha, \beta > 0$ with $\beta < \alpha$, the function ϕ satisfies*

$$\begin{aligned}\phi(\rho) &\leq A_0 \phi(R), \\ \phi(\rho) &\leq A_1 \left(\frac{\rho}{R} \right)^\alpha \phi(R) + BR^\beta\end{aligned}$$

for all $0 < \rho \leq R \leq R_0$. Then, there exists a constant $C > 0$ depending only on A_0, A_1, α, β such that

$$\phi(\rho) \leq C \left\{ \left(\frac{\rho}{R} \right)^\beta \phi(R) + B\rho^\beta \right\}.$$

Proof. Let $r < 1$ be chosen later and for $R \leq R_0$, let $\rho = rR$. Then by the assumption we have

$$\phi(rR) \leq A_1 r^\alpha \phi(R) + BR^\beta.$$

We fix $\beta < \gamma < \alpha$ and we put $r = r_0$ satisfying $A_1 r_0^\alpha \leq r_0^\gamma$. Then

$$\phi(r_0 R) \leq A_1 r_0^\alpha \phi(R) + BR^\beta \leq r_0^\gamma \phi(R) + BR^\beta.$$

Therefore, for $k \in \mathbb{N} \cup \{0\}$, we have

$$\phi(r_0^{k+1} R) \leq r_0^{(k+1)\gamma} \phi(R) + B r_0^{k\beta} R^\beta \sum_{j=0}^{k} r_0^{j(\gamma - \beta)}.$$

Since

$$\sum_{j=0}^{k} r_0^{j(\gamma-\beta)} \leq \sum_{j=0}^{\infty} r_0^{j(\gamma-\beta)} = \frac{r_0^\beta}{r_0^\beta - r_0^\gamma},$$

we obtain

$$\phi(r_0^k R) \leq r_0^{k\gamma} \phi(R) + r_0^{k\beta} \frac{BR^\beta}{r_0^\beta - r_0^\gamma}$$

for $k \in \mathbb{N}\cup\{0\}$. For $\rho > 0$, there exists $k_0 \in \mathbb{N}\cup\{0\}$ such that $r_0^{k_0+1}R \le \rho \le r_0^{k_0}R$. Therefore, we obtain

$$\phi(\rho) \le A_0\phi(r_0^{k_0}R) \le A_0\bigg(r_0^{k\gamma}\phi(R) + r_0^{k\beta}\frac{BR^\beta}{r_0^\beta - r_0^\gamma}\bigg) \le A_0\bigg(r_0^{-\gamma}\Big(\frac{\rho}{R}\Big)^\gamma\phi(R) + r_0^{-\beta}\Big(\frac{\rho}{R}\Big)^\beta\frac{BR^\beta}{r_0^\beta - r_0^\gamma}\bigg).$$

□

Proof of Theorem 1.2. Fix $0 < \rho < R < 1$. Then

(2.3)
$$\iint_{Q_\rho} |\nabla u-(\nabla u)_{Q_\rho}|^p\,dtdx \le C(p)\left(\iint_{Q_\rho}|\nabla u - \nabla v|^p\,dtdx + \iint_{Q_\rho}|\nabla v - (\nabla v)_{Q_\rho}|^p\,dtdx\right).$$

First, we estimate $\iint_{Q_\rho}|\nabla v - (\nabla v)_{Q_\rho}|^p\,dtdx$. Either if $0 < \rho \le \frac{R}{2}$, then by Lemma 2.7

$$\begin{aligned}\iint_{Q_\rho}|\nabla v - (\nabla v)_{Q_\rho}|^p\,dtdx &\le \iint_{Q_\rho}\operatorname{osc}(\nabla v)^p\,dtdx \\ &\le C(n,p)\|\nabla v\|^p_{L^\infty(Q_{\frac{R^{1-\delta}}{2}})}R^{n+2+p\delta} \\ &\quad + C(n,p)\|\nabla v\|^p_{L^\infty(Q_{\frac{R^{1-\delta}}{2}})}\Big(1+\max\{1,\|\nabla v\|^{\frac{p-2}{2}}_{L^\infty(Q_{\frac{R^{1-\delta}}{2}})}\}\Big)^{p\alpha_0}R^{n+2+p\delta\alpha_0}.\end{aligned}$$

Otherwise, if $\frac{R}{2} \le \rho \le R$, then

$$\begin{aligned}\iint_{Q_\rho}|\nabla v - (\nabla v)_{Q_\rho}|^p\,dtdx &\le 2^{n+2+p}\Big(\frac{\rho}{R}\Big)^{n+2+p}\iint_{Q_\rho}|\nabla v - (\nabla v)_{Q_\rho}|^p\,dtdx \\ &\le 2^{n+2+p}\Big(\frac{\rho}{R}\Big)^{n+2+p}\iint_{Q_R}|\nabla v - (\nabla v)_{Q_R}|^p\,dtdx.\end{aligned}$$

Since

$$\begin{aligned}&\iint_{Q_R}|\nabla v - (\nabla v)_{Q_R}|^p\,dtdx \\ &\qquad\le C(n,p)\iint_{Q_R}|\nabla v - \nabla u|^p\,dtdx + C(n,p)\iint_{Q_R}|\nabla u - (\nabla u)_{Q_R}|^p\,dtdx,\end{aligned}$$

we obtain

$$\begin{aligned}\iint_{Q_\rho}|\nabla v - (\nabla v)_{Q_\rho}|^p\,dtdx &\le C(n,p)\Big(\frac{\rho}{R}\Big)^{n+2+p}\iint_{Q_R}|\nabla u - (\nabla u)_{Q_R}|^p\,dtdx \\ &\quad + \|\nabla v\|^p_{L^\infty(Q_{\frac{R^{1-\delta}}{2}})}R^{n+2+p\delta} \\ &\quad + C(n,p)\|\nabla v\|^p_{L^\infty(Q_{\frac{R^{1-\delta}}{2}})}\Big(1+\max\{1,\|\nabla v\|^{\frac{p-2}{2}}_{L^\infty(Q_{\frac{R^{1-\delta}}{2}})}\}\Big)^{p\alpha_0}R^{n+2+p\delta\alpha_0} \\ &\quad + C(n,p)\iint_{Q_R}|\nabla u - \nabla v|^p\,dtdx.\end{aligned}$$

Next, we estimate $\iint_{Q_R} |\nabla v - \nabla u|^p \, dtdx$. By Lemma 2.1, we have

$$\iint_{Q_R} |\nabla u - \nabla v|^p \, dtdx \leq C(n,p)KR^{\gamma}.$$

Therefore, by (2.3), we obtain

$$\begin{aligned}(2.4)\quad \iint_{Q_\rho} |\nabla u - (\nabla u)_{Q_\rho}|^p \, dtdx &\leq C(n,p)\left(\frac{\rho}{R}\right)^{n+2+p} \iint_{Q_R} |\nabla u - (\nabla u)_{Q_R}|^p \, dtdx \\ &\quad + C(n,p)\|\nabla v\|^p_{L^\infty(Q_{\frac{R^{1-\delta}}{2}})} R^{n+2+p\delta} \\ &\quad + C(n,p)\|\nabla v\|^p_{L^\infty(Q_{\frac{R^{1-\delta}}{2}})}\left(1 + \max\{1, \|\nabla v\|^{\frac{p-2}{2}}_{L^\infty(Q_{\frac{R^{1-\delta}}{2}})}\}\right)^{p\alpha_0} R^{n+2+p\delta\alpha_0} \\ &\quad + C(n,p)KR^{\gamma}.\end{aligned}$$

Since $\gamma > n+2$, we can apply Lemma 2.8 for $\phi(\rho) = \iint_{Q_\rho} |\nabla u - (\nabla u)_{Q_\rho}|^p \, dtdx$ and we obtain

$$\iint_{Q_\rho} |\nabla u - (\nabla u)_{Q_\rho}|^p \, dtdx \leq C\rho^{n+2+p\gamma}$$

for some $\gamma > 0$. Therefore ∇u is Hölder continuous with exponent $\gamma > 0$. □

Remark 2.9. If the assumption (1.3) holds, then we can modify the estimate (2.4) by

$$\begin{aligned}\iint_{Q_\rho} |\nabla u - (\nabla u)_{Q_\rho}|^p \, dtdx &\leq C\left(\frac{\rho}{R}\right)^{n+2+p} \iint_{Q_R} |\nabla u - (\nabla u)_{Q_R}|^p \, dtdx \\ &\quad + CR^{n+2+p\delta - \frac{\sigma}{2}(1-\delta)p} + CR^{n+2+p\delta\alpha_0 - \frac{\sigma}{2}p(1-\delta)(1+\frac{p-2}{p}\alpha_0)} + CKR^{\gamma}.\end{aligned}$$

Therefore for sufficiently small $\sigma > 0$, we obtain

$$\iint_{Q_\rho} |\nabla u - (\nabla u)_{Q_\rho}|^p \, dtdx \leq C\left(\frac{\rho}{R}\right)^{n+2+p} \iint_{Q_R} |\nabla u - (\nabla u)_{Q_R}|^p \, dtdx + CR^{n+2+p\gamma}$$

for some $\gamma > 0$ and we can apply Lemma 2.8.

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