Proof of Theorem 1, Part 2 In this part we consider Proposition 2 with $\psi = r^p\sqrt{k_0}$ and differentiated both sides by t:

$$\frac{d}{dt}\int_{S_t}\frac{r^pk_0}{\sqrt{k}}\Big\{|\tilde{x}\cdot\theta_\sigma|^2-\frac{1}{2}|\theta_\sigma|^2\Big\}dS = \mathrm{Re}\int_{S_t}\frac{r^pk_0}{\sqrt{k}}\Big[\Big(\frac{1}{r}-\frac{p}{2r}-\frac{k_0'}{2k_0}+\frac{\partial_r k}{2k}\Big)|\theta_\sigma|^2$$

$$+\Big(-\frac{1}{r}+\frac{p}{r}+\frac{k_0'}{k_0}-\frac{\partial_r k}{2k}\Big)|\tilde{x}\cdot\theta_\sigma|^2+\frac{\tilde{\nabla}k}{2k}\cdot\theta_\sigma(\tilde{x}\cdot\overline{\theta_\sigma})+\mathcal{B}(u_\sigma,\theta_\sigma)\Big]dS$$

$$+\mathrm{Re}\int_{S_t}\frac{r^pk_0}{\sqrt{k}}\Big[2\sigma'|\tilde{x}\cdot\theta_\sigma|^2+(q_{K,\sigma}-q_K)u_\sigma(\tilde{x}\cdot\overline{\theta_\sigma})\Big]dS \equiv J_1+J_2 \qquad (4.8)$$

Proposition 0.1 *There exists $R_3 \geq R_1$ such that*

$$J_1 \geq \int_{S_t}\frac{r^pk_0}{\sqrt{k}}\frac{p-1-\beta}{r}|\tilde{x}\cdot\theta_\sigma|^2dS \quad for\ t\geq R_3.$$

Proof By means of $(K.2)$ and $(K.3)$

$$\frac{2-p}{2r}-\frac{k_0'}{2k_0}+\frac{\partial_r k}{2k}\geq\frac{2-p}{2r}-C_2(\mu),$$

$$\frac{p-1}{r}+\frac{k_0'}{k_0}-\frac{\partial_r k}{2k}\geq\frac{p-1-\beta}{r}-C_2(\mu).$$

Thus, taking also account of Lemma 3, we have for $t \geq R_1$

$$J_1 \geq \int_{S_t}\frac{r^pk_0}{\sqrt{k}}\Big[\frac{2-p}{2r}|\theta_\sigma|^2+\frac{p-1-\beta}{r}|\tilde{x}\cdot\theta_\sigma|^2-(2C_2+C_3)\mu|\theta_\sigma|^2\Big]dS,$$

and the desired inequality follows if we choose $R_3 \geq R_1$ to satisfy

$$\frac{2-p}{2r} > (2C_2+C_3)\mu(r) \ \text{ in } r\geq R_3. \qquad \square$$

Next note that

$$q_{K,\sigma}-q_K=\sigma''-\sigma'^2-2\sigma'\Big(-i\sqrt{k}+\frac{\partial_r k}{4k}\Big).$$

Then

$$2\sigma'|\tilde{x}\cdot\theta_\sigma|^2+\mathrm{Re}\{(q_{K,\sigma}-q_k)u_\sigma(\tilde{x}\cdot\overline{\theta_\sigma})\}$$

$$=\mathrm{Re}\Big\{(\sigma''-\sigma'^2)u_\sigma(\tilde{x}\cdot\overline{\theta_\sigma})+2\sigma'\Big(\tilde{x}\cdot\nabla_b+\frac{n-1}{2r}\Big)u_\sigma(\tilde{x}\cdot\overline{\theta_\sigma})\Big\}$$

$$=\mathrm{Re}\Big\{\sigma''u_\sigma\Big(\tilde{x}\cdot\overline{\nabla_b u_\sigma}+\frac{n-1}{2r}\overline{u_\sigma}\Big)\Big\}+\sigma''\frac{\partial_r k}{4k}|u_\sigma|^2-\frac{1}{2}\nabla\cdot(\tilde{x}\sigma'^2|u_\sigma|^2)$$

$$+\Big(\sigma''\sigma' - \sigma'^2\frac{\partial_r k}{4k}\Big)|u_\sigma|^2 + 2\sigma'\Big|\tilde{x}\cdot\nabla_b u_\sigma + \frac{n-1}{2r}u_\sigma\Big|^2$$

$$+\frac{1}{2}\nabla\cdot\Big(\tilde{x}2\sigma'\frac{\partial_r k}{4k}|u_\sigma|^2\Big) - \partial_r\Big(\sigma'\frac{\partial_r k}{4k}\Big)|u_\sigma|^2 \mp 2\sigma'\sqrt{k}\mathrm{Im}\{\tilde{x}\cdot\nabla_b u_\sigma\overline{u_\sigma}\}$$

$$= -\frac{1}{2}\nabla\cdot\Big\{\tilde{x}\Big(\sigma'^2 - 2\sigma'\frac{\partial_r k}{4k}\Big)|u_\sigma|^2\Big\} + \mathrm{Re}\Big\{\sigma''u_\sigma\Big(\tilde{x}\cdot\overline{\nabla_b u_\sigma} + \frac{n-1}{2r}\overline{u_\sigma}\Big)\Big\}$$

$$+2\sigma'\Big|\tilde{x}\cdot\nabla_b u_\sigma + \frac{n-1}{2r}u_\sigma\Big|^2 + \Big(\sigma''\sigma' - \sigma'^2\frac{\partial_r k}{4k} - \sigma'\frac{\partial_r^2 k}{4k} + \sigma'\frac{(\partial_r k)^2}{4k^2}\Big)|u_\sigma|^2$$

$$\mp 2\sigma'\sqrt{k}\mathrm{Im}\{\tilde{x}\cdot\nabla_b u_\sigma\overline{u_\sigma}\}.$$

We combine this and

$$0 = \frac{1}{2}\nabla\cdot(\tilde{x}\tau|u_\sigma|^2) - \mathrm{Re}\Big\{\tau\Big(\tilde{x}\cdot\nabla_b u_\sigma + \frac{n-1}{2r}u_\sigma\Big)\overline{u_\sigma}\Big\} - \frac{\tau'}{2}|u_\sigma|^2.$$

Then by using the Schwarz inequality, we obtain

$$2\sigma'|\tilde{x}\cdot\theta_\sigma|^2 + \mathrm{Re}\{(q_{K,\sigma} - q_k)u_\sigma(\tilde{x}\cdot\overline{\theta_\sigma})\}$$

$$\geq -\frac{1}{2}\nabla\cdot\Big\{\tilde{x}\Big(\sigma'^2 - 2\sigma'\frac{\partial_r k}{4k} - \tau\Big)|u_\sigma|^2\Big\} - \Big(\frac{\sigma''^2}{4\sigma'} + \frac{\tau^2}{4\sigma'}\Big)|u_\sigma|^2$$

$$+\Big(\sigma''\sigma' - \sigma'^2\frac{\partial_r k}{4k} - \sigma'\frac{\partial_r^2 k}{4k} + \sigma'\frac{(\partial_r k)^2}{4k^2}\Big)|u_\sigma|^2 \mp 2\sigma'\sqrt{k}\mathrm{Im}\{\tilde{x}\cdot\nabla_b u_\sigma\overline{u_\sigma}\}.$$

Thus, it follows that

$$J_2 \geq -\frac{1}{2}\frac{d}{dt}\int_{S_t}\frac{r^p k_0}{\sqrt{k}}\Big(\sigma'^2 - 2\sigma'\frac{\partial_r k}{4k} - \tau\Big)|u_\sigma|^2 dS + \int_{S_t}\frac{r^p k_0}{\sqrt{k}}\Big[$$

$$(\sigma'^2 - 2\sigma'\frac{\partial_r k}{4k} - \tau\Big)\Big(\frac{p}{2r} + \frac{k_0'}{2k_0} - \frac{\partial_r k}{4k}\Big)|u_\sigma|^2 - \Big(\frac{\sigma''^2}{4\sigma'} + \frac{\tau^2}{4\sigma'}\Big)|u_\sigma|^2$$

$$+\Big(\sigma''\sigma' - \sigma'^2\frac{\partial_r k}{4k} - \sigma'\frac{\partial_r^2 k}{4k} + \sigma'\frac{(\partial_r k)^2}{4k^2}\Big)|u_\sigma|^2 \mp 2\sigma'\sqrt{k}\mathrm{Im}\{\tilde{x}\cdot\nabla_b u_\sigma\overline{u_\sigma}\}\Big]dS.$$

To give a convenient estimate of J_2 we put

$$\sigma(r) = \frac{m}{1-\gamma_0}r^{1-\gamma_0}, \quad \tau(r) = r^{-2\gamma_0}\log r \quad (1/3 < \gamma_0 < \gamma) \tag{4.9}$$

where $m > 0$ is a large parameter.

Proposition 0.2 *We choose p in* (4.5) *also to satisfy*

$$2\gamma < p < 2.$$

Then there exists $R_4 \geq R_3$ and $\alpha > 0$ such that for any $m \geq 1$ and $t \geq R_4$

$$J_2 \geq -\frac{1}{2}\frac{d}{dt}\int_{S_t}\frac{r^p k_0}{\sqrt{k}}\Big(\sigma'^2 - 2\sigma'\frac{\partial_r k}{4k} - \tau\Big)|u_\sigma|^2 dS$$

$$+\int_{S_t}\frac{r^p k_0}{\sqrt{k}}\Big[\alpha m^2 r^{-1-2\gamma_0} - O(\mu)\sqrt{k} - O(r\mu^2)k - O(r^{-3\gamma_0})(\log r)^2\sqrt{k}\Big]|u_\sigma|^2 dS.$$

Proof First note that the use of Lemma 1 yields

$$\int_{S_t} \frac{r^p k_0}{\sqrt{k}} 2\sigma' \mathrm{Im}\{\sqrt{k} u_\sigma (\tilde{x} \cdot \overline{\theta_{\sigma,1}})\} dS = 2\sigma' t^p k_0 \int_{S_t} u_\sigma \tilde{x} \cdot \overline{\nabla_b u_\sigma} dS = 0.$$

Moreover, by assumptions

$$\sigma'^2 \Big(\frac{p}{2r} + \frac{k_0'}{2k_0} - \frac{\partial_r k}{4k} \Big) + \sigma'' \sigma' - \sigma'^2 \frac{\partial_r k}{4k} \geq m^2 r^{-2\gamma_0} \Big(\frac{p - 2\gamma_0}{2r} - C_2 \mu \Big),$$

$$\begin{aligned}
-2\sigma' \frac{\partial_r k}{4k} \Big(\frac{p}{2r} + \frac{k_0'}{2k_0} - \frac{\partial_r k}{4k} \Big) &\geq -2mr^{-\gamma_0} (C_1 \mu \sqrt{k})^{1/2} \Big(\frac{p - \beta}{2r} - C_2 \mu \Big) \\
&\geq \Big\{ -m^2 r^{-2\gamma_0} \frac{\epsilon}{r} + \frac{r}{\epsilon} C_1 \mu \sqrt{k} \Big\} \Big(\frac{p - \beta}{2r} - C_2 \mu \Big) \quad (\forall \epsilon > 0),
\end{aligned}$$

$$\begin{aligned}
-\frac{\sigma''^2}{4\sigma'} - \sigma' \frac{\partial_r^2 k}{4k} + 2\sigma' \Big(\frac{\partial_r k}{4k} \Big)^2 &\geq -\frac{m\gamma_0^2}{4} r^{-2-\gamma_0} - mr^{-\gamma_0} C_1 \mu \sqrt{k} \\
&\geq -\frac{m\gamma_0^2}{4} r^{-2-\gamma_0} - m^2 r^{-2\gamma_0} \frac{\epsilon}{2r} - \frac{r}{2\epsilon} (C_1 \mu \sqrt{k})^2 \quad (\forall \epsilon > 0),
\end{aligned}$$

$$-\frac{\tau^2}{4\sigma'} - \tau \frac{\partial_r k}{4k} - \tau \Big(\frac{p}{2r} + \frac{k_0'}{2k_0} - \frac{\partial_r k}{2k} \Big) - \frac{\tau'}{2}$$

$$\geq -\Big\{ \frac{1}{4m} r^{-3\gamma_0} + r^{-2\gamma_0} (C_1 \mu \sqrt{k})^{1/2} + r^{-2\gamma_0} \Big(\frac{p}{r} - C_2 \mu \Big) + r^{-1-2\gamma_0} \Big\} (\log r)^2.$$

Since $\gamma_0 < \gamma$, summarizing these inequalities, we conclude the assertion of the proposition. □

Conclusion of the proof of Part 2 It follows from Propositions 1 and 2 that for any $m \geq 1$ and $t \geq R_4$

$$\begin{aligned}
\frac{d}{dt} F_{\sigma,\tau}(t) &= \frac{d}{dt} \int_{S_t} \frac{r^p k_0}{\sqrt{k}} \Big\{ |\tilde{x} \cdot \theta_\sigma|^2 - \frac{1}{2} |\theta_\sigma|^2 + \frac{1}{2} (\sigma'^2 - 2\sigma' \frac{\partial_r k}{4k} - \tau) |u_\sigma|^2 \Big\} dS \\
&\geq \int_{S_t} \frac{r^p k_0}{\sqrt{k}} \Big[\frac{p - 1 - \beta}{r} |\tilde{x} \cdot \theta_\sigma|^2 + \alpha m^2 r^{-1-2\gamma_0} |u_\sigma|^2 \\
&\quad - \{ O(\mu) \sqrt{k} + O(r\mu^2) k + O(r^{-3\gamma}) (\log r)^2 \sqrt{k} \} u_\sigma|^2 \Big] dS. \qquad (4.10)
\end{aligned}$$

Since $\mu(r) = o(r^{-1})$, $3\tilde{\gamma} > 1$ and $k \geq C_0$, we have

$$\int_{S_t} \frac{r^p k_0}{\sqrt{k}} \Big\{ C_0^{-1/2} O(\mu) + O(r\mu^2) + C_0^{-1/2} O(r^{-3\gamma_0}) (\log r)^2 \Big\} k |u_\sigma|^2 dS$$

$$\leq t^p k_0(t) o(t^{-1}) \int_{S_t} \sqrt{k} |u_\sigma|^2 dS \leq \int_{S_t} \frac{r^p k_0}{\sqrt{k}} o(r^{-1}) |\tilde{x} \cdot \theta_\sigma|^2 dS,$$

where Lemma 2 is used to show the last inequality. Hence

$$\frac{d}{dt}F_{\sigma,\tau}(t) \geq 0 \quad \text{in } t \geq R_5 \tag{4.11}$$

if $R_5 \geq R_4$ is sufficiently large.

By assumption that the support of u is not compact, R_5 is able to satisfy

$$\int_{S_{R_5}} \left(\frac{k_0}{k}\right)^{1/2} |u|^2 dS > 0.$$

Then as we see from (4.10), $F_{\sigma,\tau}(R_5)$ goes to ∞ as $m \to \infty$. We fix a large m satisfying $F_{\sigma,\tau}(R_5) > 0$. Then (4.11) asserts that $F_{\sigma,\tau}(t) > 0$ for $t \geq R_5$.

Now, we rewrite $F_{\sigma,\tau}(t)$ as

$$\begin{aligned} F_{\sigma,\tau}(t) &= e^{2\sigma}t^p\Big\{k_0F_0(t) + \sigma'\sqrt{k_0}\mathrm{Re}\int_{S_t}\left(\frac{k_0}{k}\right)^{1/2}(\tilde{x}\cdot\nabla u)\overline{u}dS \\ &\quad + \int_{S_t}\left(\frac{k_0}{k}\right)^{1/2}\Big(\sigma'^2 - \frac{1}{2}\tau + \sigma'\frac{n-1}{2r}\Big)|u|^2dS\Big\} \\ &= e^{2\sigma}t^pk_0^{1/2}\Big[k_0^{1/2}F_0(t) + \frac{1}{2}\sigma'\frac{d}{dt}\int_{S_t}\left(\frac{k_0}{k}\right)^{1/2}|u|^2dS \\ &\quad + \int_{S_t}\left(\frac{k_0}{k}\right)^{1/2}\Big\{\sigma'^2 - \frac{1}{2}\tau - \sigma'\Big(\frac{k_0'}{4k_0} - \frac{\partial_r k}{4k}\Big)\Big\}|u|^2dS\Big]. \end{aligned} \tag{4.12}$$

In the second equality we have used the identity

$$\mathrm{Re}\int_{S_t}\left(\frac{k_0}{k}\right)^{1/2}\Big\{(\tilde{x}\cdot\nabla u)\overline{u} + \Big(\frac{n-1}{2r} + \frac{k_0'}{4k_0} - \frac{\partial_r k}{4k}\Big)|u|^2\Big\}dS = \frac{1}{2}\frac{d}{dt}\int_{S_t}\left(\frac{k_0}{k}\right)^{1/2}|u|^2dS.$$

In (4.12) $F_{\sigma,\tau}(t) > 0$, $F_0(t) \leq 0$ when $t \geq R_5$. Moreover, it follows from (4.9) that

$$\sigma'^2 - \frac{1}{2}\tau - \sigma'\Big(\frac{k_0'}{4k_0} - \frac{\partial_r k}{4k}\Big) \leq 0 \quad \text{when } r \geq R_6$$

if $R_6 \geq R_5$ is chosen sufficiently large. So, we conclude

$$\frac{1}{2}\sigma'\frac{d}{dt}\int_{S_t}\left(\frac{k_0}{k}\right)^{1/2}|u|^2dS > 0 \quad \text{for } t \geq R_6.$$

Moreover, R_6 is able to satisfy

$$\int_{S_{R_6}} \left(\frac{k_0}{k}\right)^{1/2} |u|^2 dS > 0.$$

Then

$$\int_{S_t}\sqrt{k}|u|^2dS \geq \min_{\tilde{x}\in S_t}\frac{k(t\tilde{x})}{\sqrt{k_0(t)}}\int_{S_{R_6}}\left(\frac{k_0}{k}\right)^{1/2}|u|^2dS > 0$$

for $t \geq R_6$. This and Lemma 2 with $\sigma = 0$ establish the conclusion of Theorem 1, Part 2. $\square$

For $f \in L^2(\Omega)$ let $u = R(\zeta)f$. Then $u \in L^2(\Omega) \cap H^2_{\mathrm{loc}}(\overline{\Omega})$ and, as is proved in Ikebe-Kato [//], there exists $C_1 > 0$ depending on $\mathrm{Im}\zeta$ such that

$$\|u\| + \|(e+r)^{-\alpha/2}\nabla_b u\| \leq C_1\|f\|. \tag{5.2}$$

Lemma 0.1 *Let $\alpha \leq 2$ in (A.1) and $(e+r)f \in L^2(\Omega)$. Then there exists $0 < \epsilon < 1$ depending on $\zeta \in \Gamma_\pm$ such that*

$$\int_\Omega (e+r)^\epsilon \Big\{\frac{|\nabla_b u|^2}{-c_0} + |u|^2\Big\}dx \leq C_\infty \|(e+r)f\|^2 < \infty$$

for some $C_\infty > 0$

Proof We start the proof showing

$$\int_\Omega \frac{|\nabla_b u|^2}{-c_0} dx < \infty. \tag{5.9}$$

Let $\varphi = (-c_0)^{-1}$ in (5.5). Note that $(-c_0(r))^{-1} \leq 1$ and $\dfrac{c_0'(r)}{c_0(r)^2} \leq \sqrt{C_1\mu}(-c_0)^{-3/4}$. Then since (5.2) implies

$$\liminf_{\rho\to\infty} \int_{S_\rho} |\tilde{x}\cdot\nabla_b u||u|dS = 0,$$

it follows that

$$\begin{aligned}\frac{1}{2}\int_\Omega \frac{|\nabla_b u|^2}{-c_0}dx &\leq M_0\int_\Omega\Big\{|u|^2 + |f||u| + \frac{1}{2}|u|^2\Big\}dx + \int_{\partial\Omega}|d(x)||u|^2 dS \\ &\leq C_0\|f\|^2 < \infty \ \text{ for some } C_0 > 0,\end{aligned}$$

where

$$M_0 = \max_{x\in\Omega}\Big[\max\Big\{\frac{|c-\zeta|}{-c_0}, \frac{1}{\sqrt{-c_0}}, \frac{C_1\mu}{\sqrt{-c_0}}\Big\}\Big].$$

We shall successively show

$$\int_\Omega \varphi_m\Big\{\frac{|\nabla_b u|^2}{-c_0} + |u|^2\Big\}dx < C_m\|(e+r)f\|^2 < \infty \tag{5.10}$$

$m = 1, 2, \cdots$.

The case $m = 0$ is already verified by (5.2) and (5.9). To proceed in the next step, note that

$$(e+r)^\epsilon = e^{\epsilon\log(e+r)} = \sum_{j=0}^\infty \frac{[\epsilon\log(e+r)]^j}{j!}.$$

We put $\varphi_m(r) = \sum_{j=0}^{m} \frac{[\epsilon \log(e+r)]^j}{j!}$.

The assertion

$$\liminf_{\rho\to\infty} \int_{S_\rho} \varphi_j |\nabla_b u||u| dS = 0 \quad (j = 0, 1, \cdots, m) \tag{5.11}$$

follows from assumption: The case $j = 0$ is already used to show (5.9). Assume that (5.11) holds for $j \leq m-1$. Then by means of the inequality

$$\varphi_m = \log(e+r)\Big\{\frac{1}{\log(e+r)} + \epsilon \sum_{j=0}^{m-1} \frac{(\epsilon \log(e+r))^j}{(j+1)!}\Big\} \leq \log(e+r)(1+\epsilon\varphi_{m-1})$$

we have

$$\int_{S_\rho} \varphi_m |\nabla_b u||u| dS \leq \sqrt{-c_0} \log(e+\rho) \int_{S_\rho} (1+\epsilon\varphi_{m-1}) \frac{|\nabla_b u|}{\sqrt{-c_0}} |u| dS.$$

Since $[\sqrt{-c_0}\log(e+r)]^{-1} \notin L^1(\mathbf{R}_+)$, this implies (5.11) when $j = m$.

Now we put $\varphi = \varphi_m$ in (5.4). Then since

$$\varphi_{m-1}(r) \leq \varphi_m(r), \quad \text{and} \quad \varphi_m'(r) = \frac{\epsilon\varphi_{m-1}(r)}{e+r}, \tag{5.12}$$

noting (5.11), we can use the Schwarz inequality to obtain

$$|\mathrm{Im}\zeta|^2 \int_\Omega \varphi_m |u|^2 dx \leq \int_\Omega \varphi_m |f|^2 dS + \epsilon^2 \int_\Omega \varphi_{m-1} \frac{|\nabla_b u|^2}{(e+r)^2} dx < \infty. \tag{5.13}$$

Next, we put $\varphi = \frac{\varphi_m}{-c_0}$ in (5.5). Then since

$$\varphi' = \frac{\epsilon\varphi_{m-1}}{(-c_0)(e+r)} - \frac{\varphi_m(-c_0')}{c_0^2} \leq \frac{\varphi_m}{\sqrt{-c_0}}\Big\{\frac{\epsilon}{\sqrt{-c_0}(e+r)} + \frac{\sqrt{C_1\mu}}{(-c_0)^{1/4}}\Big\}$$

applying the Schwarz inequality, letting $\rho \to \infty$ and noting (5.11), we have

$$\frac{1}{2}\int_\Omega \frac{\varphi_m}{-c_0} |\nabla_b u|^2 dx \leq M_1 \int_\Omega \varphi_m \{|u|^2 dx + |f||u| + 2|u|^2\} dx$$
$$+ \int_{\partial\Omega} \varphi_m |d(x)||u|^2 dS < \infty, \tag{5.14}$$

where

$$M_1 = \max_{x\in\Omega}\Big[\max\Big\{\frac{|c-\zeta|}{-c_0}, \frac{1}{\sqrt{-c_0}}, \frac{\epsilon}{(-c_0)(e+r)^2} + \frac{C_1\mu}{\sqrt{-c_0}}\Big\}\Big].$$

Note here

$$\int_{\partial\Omega} \varphi_m |d(x)||u|^2 dS \leq M_2 \|f\|^2$$

for some $M_2 > 0$ independent of m. Then it follows from (5.14) that

$$\int_\Omega \frac{\varphi_{m-1}}{-c_0}|\nabla_b u|^2 dx \le 4M_1 \int_\Omega \varphi_{m-1}|u|^2 dx + (M_1 + M_2)\|(e+r)f\|^2, \tag{5.15}$$

Apply this to (5.13). Then we obtain

$$(|\mathrm{Im}\zeta|^2 - 4\epsilon^2 M_1)\int_\Omega \varphi_m |u|^2 dx \le \{1 + \epsilon^2(M_1 + M_3)\}\|(e+r)f\|^2.$$

Choose ϵ so small to satisfy $|\mathrm{Im}\zeta|^2 - 4\epsilon^2 M_1 > 0$ and combine this and (5.14). Then we can let $m \to \infty$ to conclude the assertion of the lemma. □

With these preparation, we can follow the line of Eidus [8] to prove Theorem 2.

Proof of Theorem 2 Let $\{\zeta_j, f_j\} \subset \Gamma_\pm \times L^2_{(\mu_0\sqrt{-c_0})^{-1}}$ converges to $\{\zeta_0, f_0\}$ as $j \to \infty$. Since the other case is easier, we assume that $\zeta_0 = \lambda \pm i0$, $\lambda \in I$. Let $u_j = R(\zeta_j)f_k$.

(i) As is verified by Lemma 7 or 8, each u_j satisfies the radiation conditions.

(ii) $\{u_k\}$ is pre-compact in $L^2_{\mu_0\sqrt{-c_0}}$ if it is bounded in the same space. In fact, suppose that $\{u_j\}$ is bounded in $L^2_{\mu_0\sqrt{-c_0}}(\Omega)$, then by the ellipticity of the equation we can apply Rellich compactness ceriterion to show its pre-compactness in $L^2(\Omega_R)$ for any $R > R_0$. On the other hand, Lemma 5 (ii) asserts that for any $\epsilon > 0$

$$\sup_j \|u_j\|_{\mu_0\sqrt{-c_0},\Omega'_R} < \epsilon$$

if R is chosen sufficiently large.

(iii) If $u_j \to u_0$ in $L^2_{\mu_0\sqrt{-c_0}}(\Omega)$ as $j \to \infty$, then u_0 satisfies the radiation conditions with $\zeta = \zeta_0$. In fact, since $\{u_j\}$ is bounded in $L^2_{\mu_0\sqrt{-c_0}}(\Omega'_R)$, we see by Lemma 6 that

$$\{\theta_j = \nabla_b u_j + \tilde{x}K(x,\zeta_j)u_j\}$$

is also bounded in $L^2_{\varphi'\sqrt{|k|}^{-1}}(\Omega'_R)$. So, $\{\theta_j\}$ has a weekly convergent sub-sequence in the same space. Denote the limit by w, then it follows that

$$w = \nabla_b u_0 + \tilde{x}K(x, \lambda \pm i0)u_0,$$

and u_0 is concluded to satisfy the radiation conditions.

(iv) The boundedness $\{u_j\}$ is proved by contradiction. In fact, assume that there exists a subsequence, which we also write $\{u_j\}$, such that $\|u_j\|_{\mu_0\sqrt{-c_0}} \to \infty$ as $j \to \infty$. Put $v_j = u_j/\|u_j\|_{\mu_0\sqrt{-c_0}}$. Then as is explained above, $\{\zeta_j, v_j\}$ has a convergent subsequence, and if we denote the limit by $\{\lambda \pm i0, v_0\}$, then it satisfies the eigenvalue problem (2.7) and also

$$\|v_0\|_{\mu_0\sqrt{-c_0}} = 1,\ \|\partial_r v_0 + K_\pm v_0\|_{\varphi'_0\sqrt{|k|}^{-1},\Omega'_R} < \infty, \tag{5.15}$$

where $K_\pm = K(x, \lambda \pm i0)$. The second inequality implies

$$\liminf_{r\to\infty} \int_{S(r)} \sqrt{k}^{-1} |\partial_r v_0 + K_\pm v_0|^2 dS = 0$$

since $\varphi_0'(r) \notin L^1([R,\infty))$ for any $R > 0$ by Lemma 4.

Comparing this with Theorem 1, we see that v_0 has a compact support in $x \in \mathbf{R}^n$. Hence, $v_0 \equiv 0$ by the unique continuation property for solutions to (2.7). But this contradicts to the first equation of (5.15).

(v) If we apply Theorem 1 once more, then $\{u_j\}$ itself is shown to converge. □