Automorphism Groups of the Derivative Relaxations of Rank-One Generated Hyperbolicity Cones

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ICCOPT2022
July 26, 2022
Session “Riemannian Manifold Optimization and Conic Programming”

Preprint is available on arxiv. https://arxiv.org/abs/2207.11986

Classes of convex cones

Symmetric Cones (self-dual and homogeneous)

$\subsetneq$

Homogeneous Cones (Automorphism group acts transitively on interior of cone)

$\subsetneq$

Spectrahedral Cones $\{x \in \mathbb{R}^n : x_1 A_1 + \cdots + x_n A_n \succeq O\}$

$\subset$

Hyperbolicity Cones $\Lambda_+(p, e)$

Hyperbolicity cones

Hyperbolic polynomial

A homogeneous polynomial $p \in \mathbb{R}[x_1, \ldots, x_n]$ is said to be hyperbolic along direction $e \in \mathbb{R}^n$ if

- $p(e) > 0$, and
- for all $x \in \mathbb{R}^n$, the univariate polynomial $t \mapsto p(te - x)$ has all real roots.

These roots are called eigenvalues of x and denoted by

$$\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_d(x), \quad \text{where } d = \deg p.$$

Hyperbolicity cone

For p hyperbolic polynomial along e, define the hyperbolicity cone $\Lambda_{++}(p, e)$ by

$$\Lambda_{++} := \Lambda_{++}(p, e) := \{x \in \mathbb{R}^n \mid \lambda_1(x) > 0, \ldots, \lambda_d(x) > 0\}$$

We also write the closure of Λ_{++} by

$$\Lambda_{+} := \Lambda_{+}(p, e) := \{x \in \mathbb{R}^n \mid \lambda_1(x) \geq 0, \ldots, \lambda_d(x) \geq 0\}.$$

Examples

(1) The nonnegative orthant $\mathbb{R}^n_+$ is the hyperbolicity cone $\Lambda_+(p,e)$ with

$$p(x) = x_1 \cdots x_n, \quad e = (1, \dots, 1)^T$$

- $p(te - x) = (t - x_1) \cdots (t - x_n)$ has roots $\lambda_i(x) = x_i$.

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(2) The positive semidefinite cone S^n_+ is the hyperbolicity cone $\Lambda_+(p, e)$ with

$$p(x) = \det(x), \quad e = \text{the identity matrix } I_n$$

- The roots of $P(te - x) = \det(tI_n - x)$ are the eigenvalues of symmetric x.

Observation

$$\Lambda_+(p, e) = \{x : \text{roots of } t \mapsto p(te - x) \text{ are nonnegative}\}$$

- $\Lambda_+(p, e)$ is a closed convex cone.
- Generalized Lax Conjecture: Every hyperbolicity cone is a spectrahedral cone.
- Hyperbolic Programming [Güler 1997]:

$$\text{minimize } \langle c, x\rangle \text{ subject to } Ax = b,\ x \in \Lambda_+(p, e)$$

Derivative relaxation of hyperbolicity cones

For $k = 1, 2, \ldots, d = \deg p$, denote

$$D_e^k p(x) := \nabla^k p(x)[e, \ldots, e] \in \mathbb{R}[x_1, \ldots, x_n].$$

Note that $D_e^k p(x)$ is a homogeneous polynomial of degree $d - k$ (if $k < d$).

Lemma (Gårding 1959)

p is hyperbolic along $e \Longrightarrow D_e^k p(x)$ is hyperbolic along e, and

$$\Lambda_+(p, e) \subset \Lambda_+(D_e^k p, e).$$

Definition: Derivative relaxation (Renegar derivative [Renegar 2006])

The hyperbolicity cone

$$\Lambda_+^{(k)} := \Lambda_+^{(k)}(p, e) := \Lambda_+(D_e^k p, e)$$

is called the k-th derivative relaxation of $\Lambda_+(p, e)$.

$$\Lambda_+ \subset \Lambda_+^{(1)} \subset \cdots \subset \Lambda_+^{(d-1)}.$$

Examples

$$\mathbb{R}^n_+ = \Lambda_+(p, e) \text{ with } p(x) = x_1 \cdots x_n,\ \ e = (1, \ldots, 1)^T.$$

- The derivative relaxation $\mathbb{R}^{n,(k)}_+ := \Lambda_+(D^k_e p, e)$ is a hyperbolicity cone.
- $D^k_e p(x) = k!\, E_{d-k}(x)$ where $E_{d-k}(x)$ is the elementary symmetric polynomial

$$E_{d-k}(x) = \sum_{i_1 < \ldots < i_{d-k}} x_{i_1} \cdots x_{i_{d-k}}$$

of degree $d - k$.

Examples

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$$S^n_+ = \Lambda_+(p,e) \text{ with } P(x) = \det(x),\ \ e = I_n.$$

- The derivative relaxation $S^{n,(k)}_+ := \Lambda_+(D^k_e p, e)$ is a hyperbolicity cone.
- $D^k_e p(x) = k!\, E_{d-k}(\lambda(x))$.

Known facts on derivative relaxations

- [Renegar 2006]: investigation of derivative relaxations.
- $\mathbb{R}_+^{n,(k)}$ are spectrahedral cones [Sanyal 2011, Brändén 2014].
- $S_+^{n,(k)}$ are spectrahedral shadows [Saunderson and Parrilo 2015]
- $S_+^{n,(1)}$ is a spectrahedral cone [Saunderson 2018]

Spectrahedral Cones $\{x \in \mathbb{R}^n : x_1 A_1 + \cdots + x_n A_n \succeq O\}$

$\cap$

Hyperbolicity Cones $\Lambda_+(p, e)$

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Hyperbolicity Cones $\Lambda_+(p, e)$

Our focus: Geometry of derivative relaxations $\Lambda_+^{(k)}$, in particular, the structure of their automorphisms.

Automorphism groups

The automorphism group of a convex cone $K \subset \mathbb{R}^n$ is the set

$$\mathrm{Aut}(K) := \{A \in \mathrm{GL}_n(\mathbb{R}) : AK = K\}.$$

$\mathrm{Aut}(K)$ provides geometric information on K.
We can examine:

- Homogeneity of K: Every $x, y \in \mathrm{int}(K)$ admits $Ax = y$ for some $A \in \mathrm{Aut}(K)$.
- Lyapunov rank of K [Gowda & Tao 2014]: $\beta(K) :=$ the dimension of the Lie algebra of $\mathrm{Aut}(K)$.
- Perfectness of K: $\beta(K) \geq \dim K$ [Orlitzky & Gowda 2016]

Determination of $\mathrm{Aut}(K)$ is difficult for general hyperbolicity cones.
$\longrightarrow$ We focus on rank-one generated hyperbolicity cones.

Rank-one generated hyperbolicity cones

The rank of $x \in \mathbb{R}^n$ is defined by

$$\operatorname{rank}(x) = \text{"the number of nonzero eigenvalues of } x\text{"} \quad \in \{0, 1, \ldots, \deg p\}$$

- $x \in \Lambda_+(p, e)$ has full-rank (i.e., $\operatorname{rank}(x) = \deg p$) iff $x \in \operatorname{int} \Lambda_+(p, e)$.
- If x is an extreme direction of $\Lambda_+(p, e)$, i.e., $\operatorname{cone}\{x\} = \mathbb{R}_+ x$ is a face of $\Lambda_+(p, e)$, then $1 \leq \operatorname{rank}(x) < \deg p$.

Definition: Rank-one generated hyperbolicity cone

Hyperbolicity cone $\Lambda_+(p, e)$ is rank-one generated (ROG) if every extreme direction x of $\Lambda_+(p, e)$ has rank one.

- Symmetric cones (including $\mathbb{R}^n_+$ and S^n_+) are ROG hyperbolicity cones under an appropriate choice of p.

Properties of ROG hyperbolicity cones

Proposition

Let $\Lambda_+(p, e)$ be a pointed ROG hyperbolicity cone.

- p is the minimal degree polynomial defining Λ_+.
- $D_e^k p$ is the minimal degree polynomial defining $\Lambda_+^{(k)}$.

Minimal degree polynomial is unique up to a positive constant factor.
Minimalility is important to analyze the automorphisms:

Lemma

If p is a minimal degree polynomial defining Λ_+, then $A \in \mathrm{GL}_n(\mathbb{R})$ belongs to $\mathrm{Aut}(\Lambda_+)$ if and only if $Ae \in \operatorname{int} \Lambda_+$ and $p(x) = \kappa p(Ax)$ for some $\kappa > 0$.

Main Result

Theorem [I. & Lourenço, 2022]

Suppose that $\Lambda_+ = \Lambda_+(p, e)$ is pointed and ROG. Then,

$$\mathrm{Aut}(\Lambda_+^{(k)}) = \mathrm{Aut}(\Lambda_+) \cap \{A \in \mathrm{GL}_n(\mathbb{R}) : Ae = \lambda e,\ \exists \lambda > 0\}$$

holds for all $k = 1, 2, \ldots, d-3$ where $d := \deg p$.

Some consequences: Under the assumption of the theorem,

- $\mathrm{Aut}(\Lambda_+^{(1)}) = \mathrm{Aut}(\Lambda_+^{(2)}) = \cdots = \mathrm{Aut}(\Lambda_+^{(d-3)})$
- $\Lambda_+^{(1)}, \Lambda_+^{(2)}, \ldots, \Lambda_+^{(d-3)}$ are not homogeneous.
  NOTE: $\Lambda_+^{(d-2)}, \Lambda_+^{(d-1)}$ are homogeneous.

Example: Description of $\mathrm{Aut}(\mathbb{R}^{n,(k)}_+)$ for $n \geq 4$

$$\mathbb{R}^n_+ = \Lambda_+(p, e) \text{ with } p(x) = x_1 \cdots x_n,\ \ e = (1, \dots, 1)^T.$$

The derivative $\mathbb{R}^{n,(k)}_+ = \Lambda_+(D^k_e p, e)$ is the hyperbolicity cone associated with the elementary symmetric polynomial $\sum_{i_1 < \dots < i_{d-k}} x_{i_1} \cdots x_{i_{d-k}}$ of degree $d - k$.

$$\mathbb{R}^n_+ \subset \mathbb{R}^{n,(1)}_+ \subset \cdots \subset \mathbb{R}^{n,(n-3)}_+ \subset \mathbb{R}^{n,(n-2)}_+ \subset \mathbb{R}^{n,(n-1)}_+$$

- $\mathrm{Aut}(\mathbb{R}^n_+) = \{\mathrm{Diag}(d)P : d \in \mathbb{R}^n_{++},\ P \text{ permutation matrix}\}$
- $A \in \mathrm{Aut}(\mathbb{R}^{n,(k)}_+) \iff A \in \mathrm{Aut}(\mathbb{R}^n_+) \text{ and } Ae = \lambda e \ (\lambda > 0)$
  $\iff A = \mathrm{Diag}(d)P \text{ and } d_1 = d_2 = \cdots = d_n.$

Corollary

For $k = 1, \dots, n-3$, $\quad \mathrm{Aut}(\mathbb{R}^{n,(k)}_+) = \{\alpha P : \alpha > 0,\ \ P \text{ permutation matrix}\}$

- The Lyapunov rank of $\mathbb{R}^{n,(k)}_+$ is 1.
- $\mathbb{R}^{n,(k)}_+$ is not homogeneous, not perfect.

Example: Description of $\mathrm{Aut}(S_+^{n,(k)})$ for $n \geq 4$

$$S_+^n = \Lambda_+(p, e) \text{ with } p(x) = \det(x),\ e = I_n$$

We have the derivative relaxations

$$S_+^n \subset S_+^{n,(1)} \subset \cdots \subset S_+^{n,(n-3)} \subset S_+^{n,(n-2)} \subset S_+^{n,(n-1)}$$

- $\mathrm{Aut}(S_+^n) = \{\alpha L_M : \alpha > 0,\ M \in \mathrm{GL}_n(\mathbb{R})\}$, where $L_M(x) = MxM^T$
- $A \in \mathrm{Aut}(S_+^{n,(k)}) \iff A \in \mathrm{Aut}(S_+^n)$ and $A(I_n) = \lambda I_n$ $(\lambda > 0)$
  $\iff A = \alpha L_M$ and $\alpha MM^T = \lambda I_n$ $(\alpha, \lambda > 0)$.

Corollary

For $k = 1, \ldots, n-3$, $\quad \mathrm{Aut}(S_+^{n,(k)}) = \{\alpha L_M : \alpha > 0,\ M \text{ orthogonal matrix}\}$

- The Lyapunov rank of $S_+^{n,(k)}$ is $\frac{n^2-n+2}{2}$ $\left(< \dim S_+^n = \frac{n^2+n}{2}\right)$.
- $S_+^{n,(k)}$ is not homogeneous, not perfect.

Summary

Main result: If $\Lambda_+(p, e)$ is a pointed and rank-one-generated hyperbolicity cone, then

$$\mathrm{Aut}(\Lambda_+^{(k)}) = \mathrm{Aut}(\Lambda_+) \cap \{A \in \mathrm{GL}_n(\mathbb{R}) : Ae = \lambda e \ (\lambda > 0)\}$$

for $k = 1, 2, \ldots, d-3$.

- $\Lambda_+^{(k)}$ are not homogeneous.
- $\mathrm{Aut}(\mathbb{R}_+^{n,(k)})$ and $\mathrm{Aut}(S_+^{n,(k)})$ can be determined
  $\rightarrow$ non-homogeneity, imperfectness

Further questions:

- Are homogeneous cones ROG?
- Results for more general hyperbolicity cones.

Thank you for your attention!

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ArXiv Preprint:
M. I. and B. F. Lourenço, Automorphisms of rank-one generated hyperbolicity cones and their derivative relaxations, https://arxiv.org/abs/2207.11986

Key Lemma: Gårding's inequality [Gårding 1959]

For any $x_1, \ldots, x_d \in \Lambda_{++}(p, e)$, we have

$$p(x_1)^{1/d} \cdots p(x_d)^{1/d} \leq \frac{\nabla^d p(0)[x_1, \ldots, x_d]}{d!}.$$

Equality holds iff $x_1, \ldots, x_d$ are proportional modulo $\Lambda_+ \cap (-\Lambda_+)$.